3.3.2 Collinearity for the Analytic Euclidean Plane Model

The human brain is the best pattern recognizer in history.
—Heinz-Karl Winkler (1955–?)

Corollary 3.2 stated that collinearity is invariant under an isometry of a neutral plane; therefore, collinearity is invariant under an isometry of a Euclidean plane. Further, in Exercise 3.25 of the Isometry section, you have shown that collinearity is not necessarily an invariant property for a transformation of a Euclidean plane. But, a stronger result is possible with an affine transformation of the Euclidean plane; that is, an affine transformation of the Euclidean plane does not need to be an isometry for collinearity to be preserved.

Exercise 3.35. Find an affine transformation of the Euclidean plane that is not an isometry.

Proposition 3.5. Collinearity is invariant under an affine transformation of the Euclidean plane. (Video lecture at end of Isometry - Invariant Properties ***video.)***

Proof. Let A be a matrix of an affine transformation of the Euclidean plane. Assume X, Y, and Z are distinct points. Let $X' = AX$, $Y' = AY$, and $Z' = AZ$. Then $\begin{bmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ 1 & 1 & 1 \end{bmatrix} = A\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{bmatrix}$. Take the determinant of both sides of the equation, to obtain $\begin{vmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ 1 & 1 & 1 \end{vmatrix} = |A|\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$. Since A is the matrix of an affine transformation of the Euclidean plane, $\det(A)$ is nonzero. Hence, by Proposition 3.1, the distinct points X, Y, and Z are collinear if and only if the points X', Y', and Z' are collinear . Therefore, collinearity is invariant under an affine transformation of the Euclidean plane.//

This result allows us to determine a matrix equation for determining lines.

Propostion 3.6. If A is the matrix of an affine transformation of the Euclidean plane, then the image of a line l under this transformation is given by $kl' = lA^{-1}$ for some nonzero real number k where l' is the image of l.

Proof. Let A be the matrix of an affine transformation of the Euclidean plane. Assume l is a line. By Proposition 3.5, the image of l is a line, denote it by l'. For any point X in the plane, X satisfies the matrix equation $lX = 0$ if and only if X is on l. Further, since $X' = AX$, X' satisfies the matrix equation $l'X' = 0$ if and only if X' is on l'. Hence, $l'AX = 0$ if and only if X is on l. Thus, $l'AX = 0$ if and only if $lX = 0$. Since this is true for all points X, there is a nonzero real number k such that $kl'A = l$. Multiply both sides of this equation on the right by the inverse of A, to obtain $kl' = lA^{-1}$ for some nonzero real number k.//

Important Notes. The value of k depends on the particular homogeneous coordinates of a line used to express the line. Also, the value k is not unique for a matrix A, i.e., different lines may require different values of k. In some sense, the nonzero constant k is redundant, since lines in the model are an equivalence class. For example, the line [2, 3, 4] is the same line as [10, 15, 20], since 5[2, 3, 4] = [10, 15, 20].

Example. Let $A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. The lines $p'[1, 0, 5]$, $q'[-3, 10, -11]$, and $r'[3, -5, 22]$ are the respective images of the lines $p[1, 2, 3]$, $q[-1, 3, 1]$, and $r[2, -1, 5]$. Note that $(1/2)p' = pA^{-1}$, $(1/6)q' = qA^{-1}$, and $(1/3)r' = rA^{-1}$. The equations have three different values for k for the three lines, $k = 1/2$, $1/6$, and $1/3$, respectively.

Exercise 3.36. (a) Verify the above example. (b) Find the matrix of an affine transformation that maps $p[1, 2, 3]$, $q[-1, 3, 1]$, and $r[2, -1, 5]$ to $p'[1, 0, 2]$, $q'[-1, 5, -8]$, and $r'[2, -5, 13]$, respectively. *(Hint. Need to solve a system of nine equations and nine variables. You may use a calculator or computer to solve the system.)*

3.3.1 Isometry ⇔ 3.3.3 Model - Isometry for the Analytic Euclidean Plane

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