3.4.2 Translations and Rotations for the Analytic Euclidean Plane Model

In a few minutes, a computer can make a mistake so great that it would take many men many months to equal it.
—Merle L. Meacham

The investigations from the last section indicate that a translation or a rotation of the Euclidean plane is a direct isometry. What form does the matrix of an affine translation of the Euclidean plane have? Let's investigate that question. Let T_{PQ} be an affine translation of the Euclidean plane with vector $\boldsymbol{PQ} = (a, b, 0)$ and let A be the translation matrix. Let $X(x_1, x_2, 1)$ be an arbitrary point in the Euclidean plane with image X' under the translation T_{PQ}.

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23} \\ 1 \end{bmatrix}$$

Since $\boldsymbol{XX'} = \boldsymbol{PQ}$, $(x_1' - x_1, x_2' - x_2, 0) = (a, b, 0)$ for every X in the plane. Hence, $(a_{11} - 1)x_1 + a_{12}x_2 + a_{13} = a$ and $a_{21}x_1 + (a_{22} - 1)x_2 + a_{23} = b$ for every X in the plane. Since the two expressions are true for all points X, we substitute with particular values for the point X. Let $X = (0, 0, 1)$, then $a_{13} = a$ and $a_{23} = b$. Let $X = (1, 0, 1)$, then $a_{11} - 1 + a = a$ and $a_{21} + b = b$. Hence, $a_{11} = 1$ and $a_{21} = 0$. Finally, let $X = (0, 1, 1)$, then $a_{12} + a = a$ and $a_{22} - 1 + b = b$. Hence, $a_{12} = 0$ and $a_{22} = 1$. Hence,

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

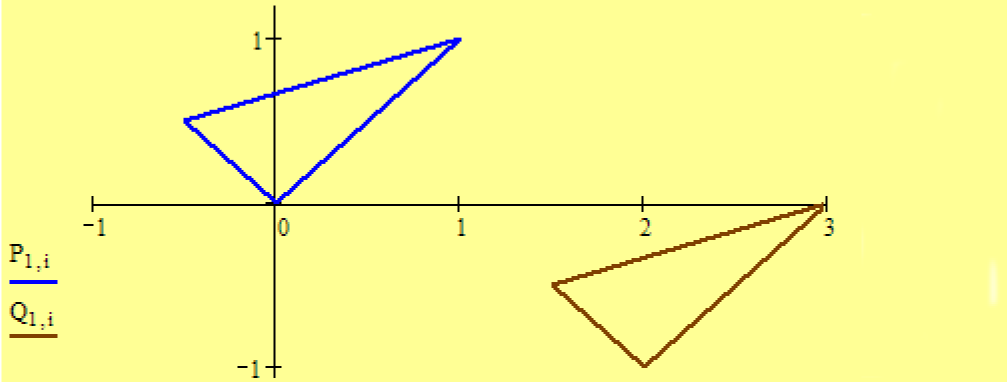
Since the det(A) = 1, a translation is a direct isometry. We summarize our results above and from the previous section with the following proposition. *(Part (b) is left for you to verify in Exercise 3.65.)*

Proposition 3.13. (a) An affine translation of the Euclidean plane is a direct isometry. (b) Any affine direct isometry of the Euclidean plane with no invariant points is a translation. (c) The matrix representation of an affine translation of the Euclidean plane T_{PQ}, defined by* vector *PQ* = (*a*, *b*, 0), *is

$$\boldsymbol{A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}}.$$

Click here to view an animation of the following example.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad |A| = 1 \qquad P = \begin{pmatrix} 0 & 1 & -0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 2 & 3 & 1.5 & 2 \\ -1 & 0 & -0.5 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



Matrix *A* is the translation matrix.
Matrix *P* is the matrix of the three endpoints of the original triangle.
Matrix *Q* = *AP* is the matrix of the translated endpoints.

Next, we investigate the form of the matrix of an affine rotation of the Euclidean plane. To simplify the problem, we first consider the problem with the center of rotation at the origin. Let $R_{O,\theta}$  be an affine rotation of the Euclidean plane with center $O(0, 0, 1)$ and let A be the translation matrix. We proceed in a similar manner as with the derivation for the translation matrix. Let $X(x_1, x_2, 1)$ be an arbitrary point in the Euclidean plane with image X' under the rotation $R_{O,\theta}$.

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23} \\ 1 \end{bmatrix}$$

Since the center of a rotation is invariant, $O(0, 0, 1)$ maps to itself; therefore, the matrix must be of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By trigonometric definitions of the circular functions, $R_{O,\theta}$ maps the point (1, 0, 1) to $(\cos\theta, \sin\theta, 1)$ and the point (0, 1, 1) to $(-\sin\theta, \cos\theta, 1)$. Hence, $a_{11} = \cos\theta$, $a_{21} = \sin\theta$, $a_{12} = -\sin\theta$, and $a_{22} = \cos\theta$. Hence, A has the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the $\det(A) = 1$, hence, A is the matrix of a direct isometry. We must show that A is the matrix of the rotation. Since A is the matrix of an isometry, $XO = X'O$ for any point X distinct from the center O. Once we show that for any point X distinct from O, $m(\angle XOX') = \theta$, by definition matrix A would be the matrix of the rotation $R_{O,\theta}$. Since $X' = AX$, we have $X'(x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, 1)$. The sides of $\angle XOX'$ are contained in the lines $\overleftrightarrow{OX}[x_2, -x_1, 0]$ and $\overleftrightarrow{OX'}[x_1\sin\theta + x_2\cos\theta, -x_1\cos\theta + x_2\sin\theta, 0]$. Substitute into the formula for the angle between two lines,

$$\tan(\angle(\overleftrightarrow{OX},\overleftrightarrow{OX'}))=\frac{x_2\left(-x_1\cos\theta+x_2\sin\theta\right)+x_1\left(x_1\sin\theta+x_2\cos\theta\right)}{x_2\left(x_1\sin\theta+x_2\cos\theta\right)-x_1\left(-x_1\cos\theta+x_2\sin\theta\right)}$$
$$=\frac{\left(x_2^2+x_1^2\right)\sin\theta}{\left(x_2^2+x_1^2\right)\cos\theta}=\tan\theta.$$

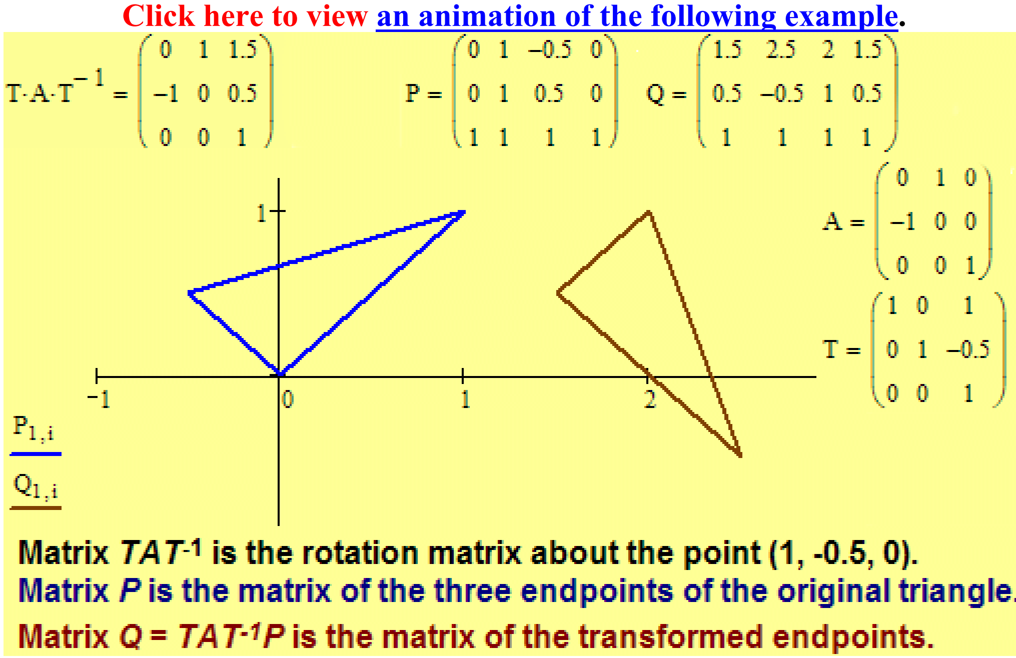
If the center of rotation, C, is at a point other than the origin O, we translate the center C to the origin, rotate about the origin, and translate back to C, i.e. $R_{C,\theta}=T_{OC}\circ R_{O,\theta}\circ T_{OC}^{-1}$. We summarize our results above and from the previous section with the following proposition. *(Part (b) and (d) are left for you to verify in Exercise 3.65.)*

Proposition 3.14. (a) An affine rotation of the Euclidean plane is a direct isometry. (b) Any affine direct isometry of the Euclidean plane with one invariant point is a rotation. (c) The matrix representation of an affine rotation of the Euclidean plane with center O(0, 0, 1) is

$$\boldsymbol{A}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(d) The matrix representation of an affine rotation of the Euclidean plane with center C not at O is $\boldsymbol{R_{C,\theta}=T_{OC}\circ R_{O,\theta}\circ T_{OC}^{-1}}$.

Click here to view an animation of the following example.



Exercise 3.57. Find a matrix of the translation that maps the point $X(3, 8, 1)$ to $Y(5, 1, 1)$ and find the image of $Z(12, 7, 1)$. Is the matrix unique?

Exercise 3.58. Find a matrix of the translation that maps the line $l[2, 3, -1]$ to $m[2, 3, 5]$. Is the matrix unique?

Exercise 3.59. Find a matrix of the rotation $R_{C,\frac{\pi}{6}}$ where $C(2, 1, 1)$ and find the image of $X(3, 6, 1)$.

Exercise 3.60. Find a direct isometry that maps $X(1, 1, 1)$ to $X'(-1, 1, 1)$ and $Y(3, 0, 1)$ to $Y'(0, 3, 1)$. Is it a translation or a rotation?

Exercise 3.61. Let the point of intersection of the lines *l*[1, –1, 0] and *m*[1, 1, –2] be the center of a rotation that maps *l* to *m*. Find the matrix of a rotation that maps *l* to *m*.

Exercise 3.62. Let the point of intersection of the lines *l*[–1, 5, 1] and *m*[3, –2, 4] be the center of a rotation that maps *l* to *m*. Find the matrix of a rotation that maps *l* to *m*.

Exercise 3.63. What is the form of the inverse of an affine translation of the Euclidean plane? Affine rotation?

Exercise 3.64. Prove the set of affine translations of the Euclidean plane is a group under matrix multiplication.

Exercise 3.65. (a) Verify part (b) of Proposition 3.13. (b) Verify part (b) of Proposition 3.14. (c) Verify part (d) of Proposition 3.14.

Exercise 3.66. Prove the set of affine rotations of the Euclidean plane with center *C* is a group under matrix multiplication.

3.4.1 Translation and Rotation 3.5.1 Reflection and Glide Reflection

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