3.5.1 Reflections and Glide Reflections

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.

—Aristotle (384–322 B.C.)

Definition. A *reflection in a line l* is a transformation of a plane, denoted R_l, such that if X is on l, then $R_l(X) = X$, and if X is not on l, then R_l maps X to X' such that l is the perpendicular bisector of $\overline{XX'}$. The line l is called the *axis* of the reflection.

Definition. A *glide reflection*, denoted G_{PQ}, is the composition of the reflection with axis $l = \overleftrightarrow{PQ}$ and the nonidentity translation T_{PQ}, i.e. $G_{PQ} = R_l \circ T_{PQ}$.

Investigation Exercises. Is each transformation an isometry? If yes, is it a direct or indirect isometry?

3.67. Draw a right triangle ΔABC with right angle at C. Accurately draw its image under each transformation.

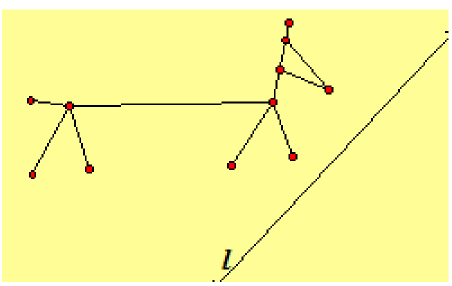
(a) R_l where $l = \overleftrightarrow{AC}$

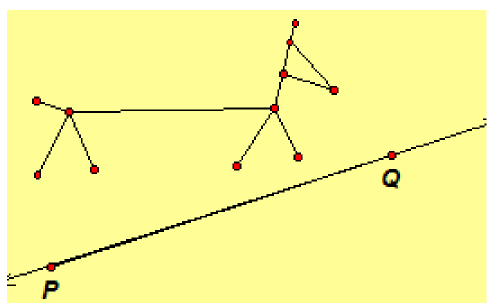
(b) R_l where $l = \overleftrightarrow{BC}$

(c) G_{AC}

(d) G_{BC}

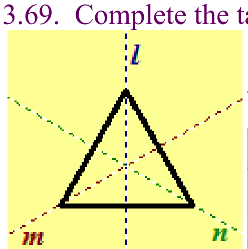
3.68. Draw the image of each transformation (a) R_l (b) G_{PQ}.





Click here to investigate dynamic illustrations of the above diagrams with GeoGebra or JavaSketchpad.

3.69. Complete the table of the compositions of symmetries for an equilateral triangle. *An animation sketch is available for Geometers Sketchpad in Geometers Sketchpad and GeoGebra Prepared Sketches and Scripts.*

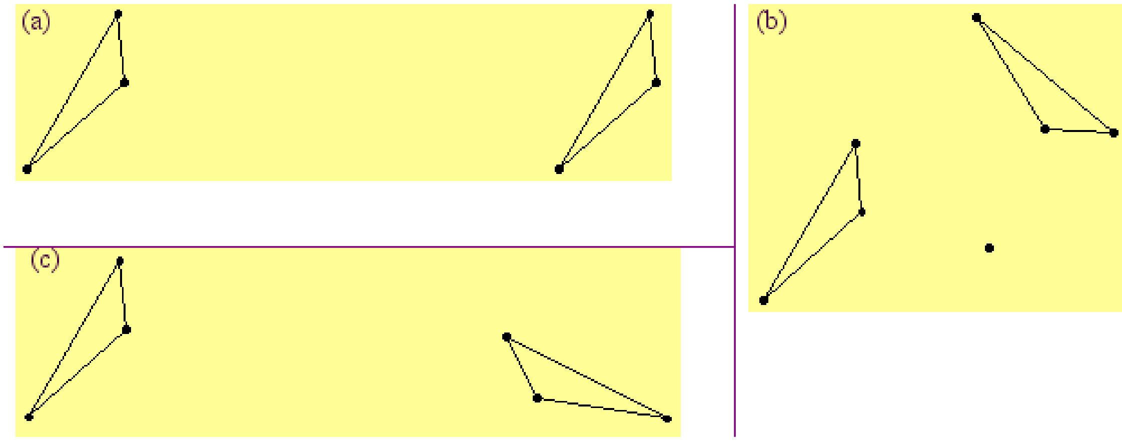


	$I = R_{O,0}$	$R_{O,120}$	$R_{O,240}$	R_l	R_m	R_n
$I = R_{O,0}$						
$R_{O,120}$						
$R_{O,240}$						
R_l						
R_m						
R_n						

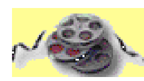
Is the set of symmetries of an equilateral triangle a group? Explain.

3.70. Complete a table of the compositions of the symmetries for a square. Is the set of symmetries of a square a group? Explain.

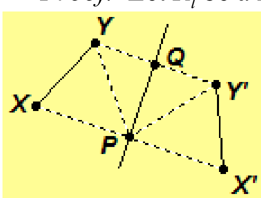
3.71. For each diagram, draw the axes of a composition of reflections that map one triangle onto the other triangle. What is the fewest number of axes of reflection that can be drawn?



Theorem 3.19. A reflection of a neutral plane is an isometry.



Proof. Let R_l be a reflection of a neutral plane. Let X and Y be any two distinct points with $X' = R_l(X)$ and $Y' = R_l(Y)$. We need to show $XY = X'Y'$. One of the following is true: (1) X and Y are on the same side of l. (2) X and Y are on opposite sides of l. (3) One of X and Y are on l. (4) Both X and Y are on l. We prove case 1 and leave the other cases as an exercise.



Assume X and Y are on the same side of l. By definition of the reflection R_l, l is the perpendicular bisector of segments $\overline{XX'}$ and $\overline{YY'}$. Hence, there are points P and Q on l such that $\overline{XP} \cong \overline{X'P}$, $\overline{YQ} \cong \overline{Y'Q}$, $\angle YQP \cong \angle Y'QP$, and $\angle XPQ \cong \angle X'PQ$. Thus, since $\overline{PQ} \cong \overline{PQ}$, we have $\Delta YQP \cong \Delta Y'QP$ by SAS. Hence, $\overline{YP} \cong \overline{Y'P}$ and $\angle YPQ \cong \angle Y'PQ$. By angle subtraction, $\angle XPY \cong \angle X'PY'$. Hence, by SAS, $\Delta XPY \cong \Delta X'PY'$. Therefore, by the definition of congruent triangles and congruent segments, $XY = X'Y'$. The proofs that $XY = X'Y'$ for the other cases is left as an exercise. Hence the reflection R_l is an isometry.//

Theorem 3.20. The inverse of a reflection of a neutral plane is the reflection itself.

Theorem 3.21. Every point on the axis of reflection is invariant under a reflection of a neutral plane.

Theorem 3.22. All lines perpendicular to the axis of reflection are invariant under a reflection of a neutral plane.

Theorem 3.23. For any two distinct points X and Y in a neutral plane, there exists exactly one reflection that maps X to Y.

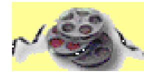
Proof. Let X and Y be two distinct points in a neutral plane. There exists a unique perpendicular bisector l of the segment $\overline{XY}$. Hence, by the definition of a reflection, R_l is the unique reflection that maps X to Y.//

Theorem 3.24. A glide reflection of a Euclidean plane is an isometry.

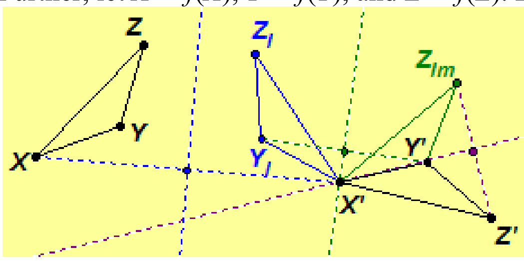
Theorem 3.25. The axis of a glide reflection is the only invariant line under a glide reflection of a Euclidean plane.

Theorem 3.26. A glide reflection of a Euclidean plane has no invariant points.

Theorem 3.27. Every isometry of a Euclidean plane is the composition of at most three reflections.
(For a dynamic diagram, GeoGebra or JavaSketchpad.)



Proof. Let f be an isometry of a Euclidean plane, and let X, Y, and Z be any three noncollinear points. Further, let $X' = f(X)$, $Y' = f(Y)$, and $Z' = f(Z)$. By Corollary 3.10, we only need to find a composition of three reflections that maps X, Y, and Z to X', Y', and Z', respectively. If X and X' are distinct, then by Theorem 3.23 there is a unique reflection R_l that maps X to X'. Let $Y_l = R_l(Y)$ and $Z_l = R_l(Z)$. Similarly, if Y_l and Y' are distinct, there is a unique reflection R_m that maps Y_l to Y'. Since f and R_l are isometries, $X'Y' = XY = X'Y_l$. Hence, X' is on line m the perpendicular bisector of $\overline{Y'Y_l}$ and $X' = R_m(X')$. Thus, the composition $R_m \circ R_l$ maps X to X', Y to Y', and Z to some point Z_{lm}. As before, if Z_{lm} and Z' are distinct, there is a unique reflection R_n that maps Z_{lm} to Z'. Since f, R_l, and R_m are isometries, $X'Z' = XZ = X'Z_l = X'Z_{lm}$ and $Y'Z' = YZ = Y_lZ_l = Y'Z_{lm}$. Hence, X' and Y' are on line n the perpendicular bisector of $\overline{Z'Z_{lm}}$, $X' = R_n(X')$ and $Y' = R_n(Y')$. Thus, the composition $R_n \circ R_m \circ R_l$ maps X to X', Y to Y', and Z to Z'. If $X = X'$, $Y_l = Y'$, or $Z_{lm} = Z'$, then omit R_l, R_m, or R_n, respectively, from the composition. By Theorem 3.20, the identity transformation is the composition of a reflection with itself. Therefore, every isometry of a Euclidean plane is the composition of no more than three reflections.//



Theorem 3.28. Every isometry of a Euclidean plane is a translation, rotation, reflection, or glide reflection.

Exercise 3.72. Repeat Exercise 3.71 by following the method given by the proof of Theorem 3.27.

Exercise 3.73. Complete the proof of Theorem 3.19 for the other cases.

Exercise 3.74. Prove Theorem 3.20.

Exercise 3.75. Prove Theorem 3.21.

Exercise 3.76. Prove Theorem 3.22.

Exercise 3.77. Prove Theorem 3.24.

Exercise 3.78. Prove Theorem 3.25.

Exercise 3.79. Prove Theorem 3.26.