3.7 Other Affine Transformations of the Euclidean Plane

Mathematicians are like lovers…Grant a mathematician the least principle, and he will draw from it a consequence which you must grant him also, and from this consequence another.
—Bernard de Fontenelle (1657–1757)

This section is an exploration section. The reader is asked to investigate various affine transformations that were not covered in previous sections and to conjecture general properties. Hopefully, the reader does not just accept the proposed conjectures, but proves or attempts to prove them. The reader should note that the matrices for isometries and similarities had certain positions which had the same value, $|a_{11}| = |a_{22}|$ and $|a_{12}| = |a_{21}|$. What happens when these values are not the same?

Definition. An *affine transformation of the Euclidean plane, T,* is a mapping that maps each point X of the Euclidean plane to a point $T(X)$ of the Euclidean plane defined by $T(X) = AX$ where $\det(A)$ is nonzero and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \text{ where each } a_{ij} \text{ is a real number.}$$

May use computer software or calculators to aid in the investigations that follow.

Investigation Exercise 3.108. Suppose $A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where a is neither 0 nor 1. Given a square $PQRS$ with vertices $P(1, 1, 1)$, $Q(-1, 1, 1)$, $R(-1, -1, 1)$, and $S(1, -1, 1)$.

a. Find the image of the square for different values of a such as –3, –1/2, 1/2, 3, etc.
b. Describe how the transformation changed the square *PQRS*.
c. Are there any invariant points or lines? If yes, what are they?
d. Let $A = \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the image of the square *PQRS* with vertices $P(3, 4, 1)$, $Q(1, 4, 1)$, $R(1, 2, 1)$, and $S(3, 2, 1)$, then repeat parts (b) and (c).
e. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the image of the square *PQRS* with vertices $P(1, 1, 1)$, $Q(-1, 1, 1)$, $R(-1, -1, 1)$, and $S(1, -1, 1)$ for different values of a not 0 or 1, then repeat parts (b) and (c).
f. These are examples of what is called a *shear*. Write a general definition of a *shear*.
g. From your definition derive one or more matrices for a shear. Write any relationships you find between shears and isometries.
h. Write any properties you find for shears.

Investigation Exercise 3.109. Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where a is not –1, 0, or 1. Given a square *PQRS* with vertices *P*(1, 0, 1), *Q*(0, 1, 1), *R*(–1, 0, 1), and *S*(0, –1, 1).

a. Find the image of the square for different values of a such as –3, –1/2, 1/2, 3, etc.
b. Describe how the transformation changed the square *PQRS*.
c. Are there any invariant points or lines? If yes, what are they?
d. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix}$. Find the image of the square *PQRS* with vertices *P*(3, 3, 1), *Q*(2, 4, 1), *R*(1, 3, 1), and *S*(2, 2, 1), then repeat parts (b) and (c).
e. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the image of the square *PQRS* with vertices *P*(1, 0, 1), *Q*(0, 1, 1), *R*(–1, 0, 1), and *S*(0, –1, 1). for different values of a not –1, 0, or 1, then repeat parts (b) and (c).
f. These are examples of what is called a *strain*. Write a general definition of a *strain*.
g. From your definition derive one or more matrices for a strain.
h. Write any relationships you find between strains and isometries.
i. Write any properties you find for strains.

Investigation Exercise 3.110. Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{21} & a_{13} \\ a_{21} & a_{11} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $c = \dfrac{a_{11}a_{12} + a_{21}a_{22}}{a_{11}^2 + a_{21}^2}$ and $d = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}^2 + a_{21}^2}$.

a. Verify the computation.
b. Write a theorem that the computation implies.

3.6.2 Model - Similarity Transformation for the Analytic Euclidean Plane

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