

Readings for Lecture/Lab 1 – Sets and Whole Numbers

How are the numerals in the statement that follows used differently? Do they represent different types of numbers?

A class of MDEV 102 has 19 students and is held on the 2nd floor of Bridges in room number 269.

The sentence has four numerals representing numbers used in three different ways.

- The 102 and 269 are numerals called *identification* or *nominal numbers* since they are used to identify, name, or label.
- The 19 is a numeral for a *cardinal number* since it answers the question: How many elements are in a set?
- The 2 is a numeral for an *ordinal number* since it specifies a position in an ordering of a group of objects.

Definitions. A *number* is the idea or abstract notion of quantity.

A *numeral* is the symbol(s) used to represent a number.

Example. The numeral 19 represents the idea that the number of students in the class is nineteen.

Our concern in this chapter is with whole numbers used as cardinal numbers in solving problems. We begin by developing these concepts, which will be used throughout the course.

The Language of Sets—Basic Terminology

Definition. A *set* is a well-defined collection of objects or ideas.

What does it mean for a set to be *well-defined*?

A set is well-defined, if there is no ambiguity as to whether or not an object belongs to it.

Examples. Which of the following are sets?

1. The collection of good students

The collection is *not* a set since the word “good” is ambiguous and the phrase is not well-defined.

2. The collection of students at MSUM with a grade point average above 3.0.

It is fairly clear which students would belong to this collection, hence, it is a set since it is well-defined.

3. The collection of young people that are residents of Minnesota
The collection is *not* a well-defined set since “young” is an ambiguous term.
4. The collection of residents of Minnesota between the ages of 18 and 25 years old
It is fairly clear who would belong to this collection, hence, it is a well-defined set.

We have three ways of describing sets:

1. by name or verbal description of the elements of a set,
2. by roster (list) form by listing the elements separated by commas and using braces to enclose the list, or
3. by set-builder notation that uses a variable and a rule to describe the elements of a set.

Examples.

Let A represent the collection of states that border Minnesota.

$$A = \{\text{North Dakota, South Dakota, Iowa, Wisconsin, Michigan}\}$$

$$A = \{x : x \text{ is a state bordering Minnesota}\}$$

This is read as “ A is the set of all x such that x is a state bordering Minnesota”.

Let B represent the collection of counting numbers.

$$B = \{1, 2, 3, 4, \dots\}$$

$$B = \{x : x \text{ is a counting number}\}$$

Definition. An object or idea in a set is called an *element (member)* of the set. The symbol \in is used to denote that an element is a member of a set and \notin is used to denote that an object is not a member of a set.

Examples. For the sets A and B used in the previous example, we have

Iowa $\in A$, Alabama $\notin A$, 8 $\in B$, and 0 $\notin B$.

Definition. The *null set (empty set)* is a set that has no members. The symbol ϕ is used to represent the null set (empty set).

Example.

$$\phi = \text{The collection of people attending MSUM who are 200 years old}$$

$$\phi = \{ \}$$

$$\phi = \{x : x \text{ is a person attending MSUM who is 200 years old.}\}$$

Note that $\{\phi\}$ does **not** symbolize the empty set; it represents a collection of empty sets.

Definition. Two sets are *equal*, if they have exactly the same elements.

Examples.

Consider the sets $A = \{a, b, c\}$, $B = \{b, c, a\}$, $C = \{a, a, c, b, c\}$, $D = \{a, b, d\}$, then $A = B = C \neq D$. The sets A , B , and C are all equal since they each have the elements a , b , and c , and no other elements. The set D is not equal to the other three sets since $c \in A$ but $c \notin D$.

Note that the order of the elements does not matter and an element does not need to be listed more than once.

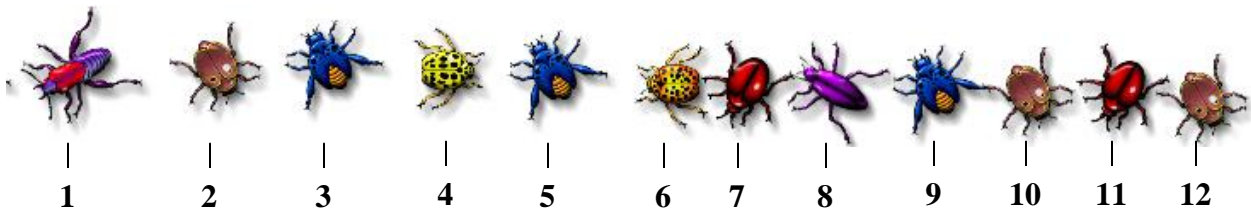
Sets and Whole Numbers

The first number set we learned as children was the set of *natural* or *counting numbers*. This is also true in the history of humans; early humans first counted the objects around them.

Example. How many insects are illustrated below?



A small child would point at each insect and count them.



The child would say there are 12 insects since the child has set up a 1-1 correspondence with the set $\{1, 2, 3, \dots, 12\}$ and the set of insects. Now we define these connections between sets and counting numbers.

Definition. If each element of a set A can be paired with exactly one element of a set B and if each element of B can be paired with exactly one element of A , then there is a *one-to-one correspondence* between A and B . The sets A and B are said to be *equivalent*, denoted as $A \sim B$.

Example.

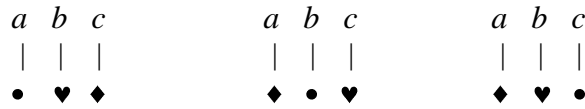
Show set $A = \{a, b, c\}$ and set $B = \{\bullet, \blacklozenge, \heartsuit\}$ are equivalent, i.e., show $A \sim B$.

a	b	c
•	◆	♥

The two sets are equivalent since a one-to-one correspondence can be made between the two sets.

Note that $A \sim B$, but $A \neq B$.

We illustrate that there are other 1-to-1 correspondences that could have been made to show the sets are equivalent.



How many other distinct one-to-one correspondences could be made where a, b, c are kept in the same order? What are they? That is, how many different one-to-one correspondences could be made?

Important Note. Equal sets are equivalent, but equivalent sets may *not* be equal. This was illustrated in the above example where $A \sim B$, but $A \neq B$. Two sets are equal when they have exactly the same elements, and sets are equivalent when a one-to-one correspondence can be set up between the two sets.

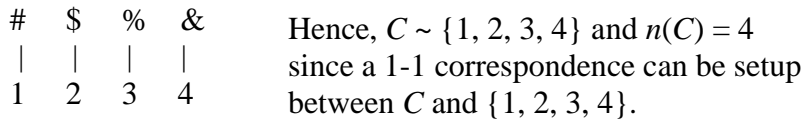
We have shown a close relationship between the concept of one-to-one correspondence and the idea of the number of elements in a set, called the cardinality of a set. (See the counting of the insects above.) This exploration has led us to the following definitions relating the sets of natural and whole numbers to finite sets. Further, we note that this relationship is closely related to how small children learn to count.

Definitions. The set of *natural numbers* (or *counting numbers*) is the set $\{1, 2, 3, \dots\}$.
The set of *whole numbers* is the set $\{0, 1, 2, 3, \dots\}$.

Definition. Let A be a finite set. If set A is equivalent to the subset $\{1, 2, 3, \dots, N\}$ of the natural numbers, we say that the *cardinal number* for set A is N . Notation: $n(A) = N$. The cardinal number for the empty set, ϕ , is 0, i.e., $n(\phi) = 0$.

Example.

Let $C = \{\#, \$, \%, \&\}$. Show $n(C) = 4$.



You probably learned the cardinal number zero, 0, much later in life, well after you learned how to count. This is also true in the history of humans. The cardinal number zero was invented much later than the natural numbers.

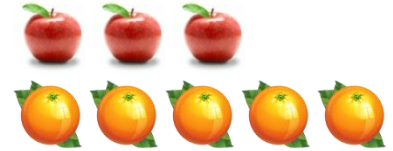
Additional Notes. We will not give precise definitions for the terms *finite* and *infinite*. We will consider a *finite* set to be a set that has a cardinal number that is a natural number and an *infinite* set as a set that is not finite. Think of a finite set as a set that has a limited number of elements and an infinite set as a set that has an unlimited number of elements.

Side Note. The cardinal number for any set equivalent to the set of all the natural numbers is \aleph_0 , read as aleph-nought. Aleph is a letter in the Hebrew alphabet.

Ordering the Whole Numbers

What do we mean by $3 < 5$? We illustrate with an example.

The three apples can only be set up with a 1-1 correspondence with three of the oranges. Two of the oranges cannot be paired with any apples. We say that the set of apples is equivalent to a subset of the set of oranges.



The above example motivates the following definitions.

Definition. A set A is a *subset* of a set B , denoted $A \subseteq B$, if every element of A is also an element of B . The set A is a *proper subset* of set B , denoted $A \subset B$, if A is a subset of B and A is not equal to set B (written symbolically as $A \subset B$ if $A \subseteq B$ and $A \neq B$).

Examples.

1. Let $A = \{1, 3, 5\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. Then $A \subseteq B$ and $A \subset B$ since each element of A is an element of B and 2 is an element of B but 2 is not an element of A .
2. Let $C = \{a, b, c\}$ and $D = \{b, c, d, e\}$. Then $C \not\subseteq D$. C is not a subset of D since D does *not* contain the element a of set C .
3. $A \subseteq A$ and $\emptyset \subseteq A$. (A set is a subset of itself, and the empty set is a subset of every set.)

Definition. Let $a = n(A)$ and $b = n(B)$ be whole numbers for finite sets A and B . If A is equivalent to a proper subset of B , then $a < b$ or $b > a$, these are read as *a is less than b* and *b is greater than a* .

Example. Consider our earlier example with the apples and oranges. Since the set of apples is equivalent to a proper subset of the set of oranges, we have that $3 < 5$.