## Session 1 – Motivation for Sets and Whole Numbers

***How are the numerals in the statement that follows used differently? Do they represent different types of numbers?***

> *A class of MDEV 102 has 19 students and is held on the 2nd floor of Bridges in room number 269.*

The sentence has four numerals representing numbers used in three different ways.

- The 102 and 269 are numerals called *identification* or *nominal numbers* since they are used to identify, name, or label.
- The 19 is a numeral for a *cardinal number* since it answers the question: How many elements are in the class of students (group, collection, set)?
- The 2 is a numeral for an *ordinal number* since it specifies a position in an ordering of a group of objects.

***Number and Numeral:*** A *number* is the idea or abstract notion of quantity.
A *numeral* is the symbol(s) used to represent a number.

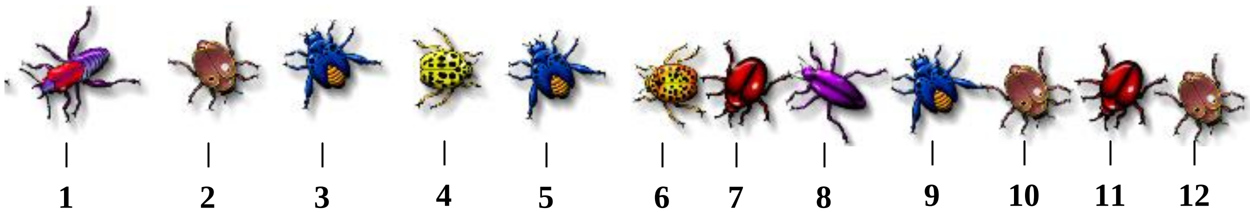
Example: In the above example, the numeral 19 represents the number (quantity) of students in the class. The numeral 102 represents the number (identity) of the course. The numeral 2 represents the number (position) of the floor. The numeral 269 represents the number (location) of the room.

Analogy: The difference in meaning between a number and a numeral is like the difference between you and your name. Your name represents you, but it is not who you are as a person. Similarly, a numeral represents a number, but it is not the value.

The purpose of this session is to motivate the basic concepts that will be used throughout this chapter and the course. Our concern in this chapter is with whole numbers used as cardinal numbers in solving problems. In the next section, we will begin the formal development of the terminology we will use. But first, we illustrate how the concepts have humble beginnings with early humans and with small children.

The first number set we learned as children was the set of *natural* or *counting numbers*. This is also true in the history of humans; early humans first counted the objects around them. Observe a small child (maybe a three year old) when the child is learning to count. What would the child do when counting the collection of insects illustrated below?

Most small children would point at each insect and say each number as the child counts the insects. We illustrate below by drawing a correspondence between the insects and the numerals 1 through 12.




After finishing the child may say that there are 12 insects.

This example indicates there is a correspondence between the collection of insects and the first twelve counting numbers. This motivates the introduction of the two concepts: set and one-to-one correspondence.

***Set*:** In mathematics, we call collections of objects *sets*.

In the next session, a more carefully formed definition will be given for a collection of objects to be a set. Also, the proper notation used for sets will be given. This proper notation will be used in this course and is used in other mathematics courses. Though, for this introduction, this informal definition is good enough.

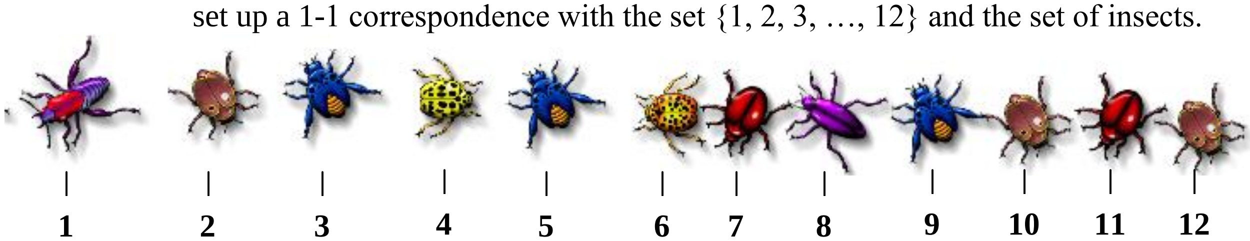
Example: The collection of insects above is a set of insects.

Example: The collection of numerals that represents the first twelve counting numbers is the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

***One-to-One Correspondence and Equivalence of Sets:*** If the elements of two sets can be paired so that each element is paired with exactly one element from the other set, then there is a *one-to-one correspondence* between the two sets. The two sets are said to be *equivalent*.

Notation: The sets $A$ and $B$ are *equivalent* and is denoted as $A \sim B$.

Example: From the earlier example, the child would say there are 12 insects since the child has set up a 1-1 correspondence with the set {1, 2, 3, …, 12} and the set of insects.




Example: Show set $A = \{a, b, c\}$ and set $B = \{\bullet, \blacklozenge, \heartsuit\}$ are equivalent, i.e., show $A \sim B$.

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| \| | \| | \| |
| $\bullet$ | $\blacklozenge$ | $\heartsuit$ |

The two sets are equivalent since a one-to-one correspondence can be made between the two sets. Note that $A \sim B$, but $A \neq B$.

***Note:*** Here equal and equivalent mean two different things. Equal sets are equivalent, but equivalent sets may *not* be equal. This was illustrated in the above example where $A \sim B$, but $A \neq B$. Two sets are equal when they have exactly the same elements, and sets are equivalent when a one-to-one correspondence can be set up between the two sets.

We have shown a close relationship between the concept of one-to-one correspondence and the idea of the number of elements in a set, called the cardinality of a set. (See the counting of the insects above.) This exploration has led us to the following definitions relating the sets of natural and whole numbers to many other sets. Further, we note that this relationship is closely related to how small children learn to count.

***Sets of Numbers:*** The set of *natural numbers* (or *counting numbers*) is the set $\boldsymbol{N} = \{1, 2, 3, \ldots\}$.
The set of *whole numbers* is the set $\boldsymbol{W} = \{0, 1, 2, 3, \ldots\}$.

***Cardinal Number of a Set*:** The number of elements in a set is the *cardinal number* of that set.

Notation: If a set $A$ is equivalent to the set $\{1, 2, 3, \ldots, N\}$, we write $n(A) = N$ and say "The cardinal number of set $A$ is $N$."
Also, $n(\varnothing) = 0$. The cardinal number for an empty set is zero.

Note that this gives us a relationship between the whole numbers and many other sets.

Example: When we counted the insects in the above example, we have shown a one-to-one correspondence between the set $\{1, 2, 3, \ldots, 12\}$ and the set of insects, i.e. we showed the set of insects is equivalent to the set $\{1, 2, 3, \ldots, 12\}$.We showed the two sets are equivalent. This means that when we said there were twelve insects, we were saying that the cardinal number for the set of insects was 12.

More examples for the cardinal numbers for sets will be given in the next session.

You probably learned the cardinal number zero, 0, much later in life, well after you learned how to count. This is also true in the history of humans. The cardinal number zero was invented much later than any of the natural numbers.

***One-to-One Correspondence and Word Problems***

We can think of a 1-1 correspondence as "an ordering of the selections" from two groups of objects. In some applications we may need or want to know: "How many different ways assignments can be made?" or "How many different ways objects can be ranked in an order?"

Example: Cary, Dana, and Pat are elected to be president, secretary, and treasurer. What are the possible office assignments? How many assignments are possible? We could elect Cary as president, Dana as secretary, and Pat as treasure, but there are other possibilities.

*Solution*: We have two sets: the three people - {Cary, Dana, Pat} - and the three offices - {president, secretary, treasurer}. The assignment of each of the people to a particular office where each person can hold only one office is a 1-1 correspondence between the set of people and the set of offices. The number of possibilities is the number of ways a 1-1 correspondence can be formed. Here is an organized way to list them:

| | | |
|---|---|---|
| Cary – President<br>Dana – Secretary<br>Pat – Treasurer | Cary – Secretary<br>Dana – President<br>Pat – Treasurer | Cary – Treasurer<br>Dana – President<br>Pat – Secretary |
| Cary – President<br>Dana – Treasurer<br>Pat – Secretary | Cary – Secretary<br>Dana – Treasurer<br>Pat – President | Cary – Treasurer<br>Dana – Secretary<br>Pat – President |

We have six ways of making the assignments. Further, note that $6 = 3 \cdot 2 \cdot 1$. This result follows from the fact that we have three choices for Cary, once that choice was made there are two choices for Dana, and once that choice is made there is only one choice remaining for Pat. This idea of multiplying the number of choices for each to find the total number of possibilities will be covered later in the course in Session 7 with the Fundamental Counting Principle.

Here is another way to illustrate the solution by just using the first letter of each name. We have two sets: people {C, D, P} and offices {p, s, t}. Here is an organized way to list the six possible office assignments:

| | | | | | |
|---|---|---|---|---|---|
| C D P | C D P | C D P | C D P | C D P | C D P |
| p s t | p t s | s p t | s t p | t p s | t s p |