Session 5 – More Set Terminology

How would you solve this problem? Try to solve it.

> *A group of 30 students went on a camping trip. Of these, 12 return with both sunburns and insect bites, and 20 report sunburn. How many suffered only insect bites if it is known that only three students suffered neither?*

Some people enjoy problems of this type as fun puzzles; whereas, others find them to be a real headache. People who recognize and are able to solve problems of this type often obtain more information from data given in business meetings or when reading news reports than other people. This may lead to better business and personal decisions. Further, it is one type of problem often found on gateway exams such as MCAT (medical school), LSAT (law school), ASVAB (military), GRE (graduate school), and the Civil Service Exam (federal employment). The intention of these types of questions is to determine if a person is able to think logically. In this lesson, we will be learning a method for visualizing and solving such problems. But first, we need to learn a few more terms involving sets that can aid us in solving problems.

We will come back to the above problem later and solve it with an approach that uses set concepts.

Universal Set and Venn Diagrams

Universal Set: A *universal set*, sometimes called the *universe*, is the set of <u>all</u> items under consideration for a particular problem or situation. We will let set U, unless otherwise defined, represent the universe in a given problem or situation. We always need to be aware of the specific universal set used in any problem. The universal set chosen can make a huge difference in the answer to a problem.

Example: In the opening problem above, the universal set is the "set of the thirty students that went on the camping trip."

Example: Find the solution set for the equation, $x + 5 = 3$.

For the universal set $\boldsymbol{N}$ = the set of natural numbers = $\{1, 2, 3, \ldots\}$, there is no natural number solution to this equation, i.e., the solution set is the empty set.

For the universal set U = the set of integers = $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the -2 satisfies the equation since $-2 + 5 = 3$. So, the solution set is $\{-2\}$.

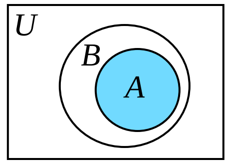
Note: The problem has a different solution set depending on the universe used, $\emptyset$ or $\{-2\}$.

Venn Diagrams: A *Venn diagram* is a drawing that can be used to show how sets are related.

A rectangle is used to represent the universal set (universe). (Remember that the universe is everything under consideration for the given problem.)

Circles are used to represent subsets of the universe, i.e., sets that exist within the universe. The relative placement of the circles shows how those sets are related. In the last session, we worked with the concept of subsets.

A Venn diagram of $A \subseteq B$ looks like the following:



The diagram shows that every element in set A is also in set B.

Set Operations – Union and Intersection

Operations like addition, subtraction, multiplication, and division are numeric operations. These numeric operations work with numbers and produce numeric solutions.

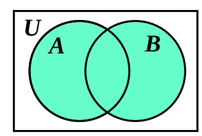
Example: The equation $4 + 3 = 7$ illustrates a numeric operation where the operation is addition.

In contrast, operations with sets (set operations) work with sets and have sets for answers. The set operations we will work with in this session are union and intersection.

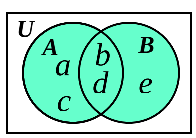
Union: The set operation *union* brings together all of the elements of both sets whether those elements are in one or both of the sets.

In set-builder notation, $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$.

The Venn diagram for $A \cup B$ is shown to the right where the shaded region represents the set $A \cup B$.



Example: Let $A = \{a, b, c, d\}$ and $B = \{b, d, e\}$. Then $A \cup B = \{a, b, c, d, e\}$. The Venn diagram illustrates the result.



The elements b and d are not written twice in the union even though they are in both sets. Remember that in the roster notation for sets we do not repeat elements within the set braces.

Example: On campus, why is *union* used in the name of the Comstock Memorial Union (CMU)? The CMU is a place where all members of the campus may gather or come together, a union of all groups (sets) of students, faculty, and staff on campus.

Example: Let $G = \{t, a, n\}$ and $H = \{n, a, t\}$. Then $G \cup H = \{a, n, t\}$.
Note that here $G = H = G \cup H$.

Example: Let $C = \{2, 6, 10, 14, \ldots\}$ and $D = \{2, 4, 6, 8, \ldots\}$.
Then $C \cup D = \{2, 4, 6, 8, \ldots\} = D$.

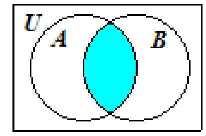
Example: Let $E = \{d, a, y\}$ and $F = \{n, i, g, h, t\}$. Then $E \cup F = \{d, a, y, n, i, g, h, t\}$.

Note: In all the examples, each set forming the union is a subset of the union, i.e. $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

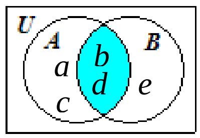
Intersection: The set operation *intersection* takes only the elements that are in both sets. The intersection contains the elements that the two sets have in common. The intersection is where the two sets overlap.

In set-builder notation, $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$.

The Venn diagram for $A \cap B$ is shown to the right where the shaded region represents the set $A \cap B$.



Example: Let $A = \{a, b, c, d\}$ and $B = \{b, d, e\}$. Then $A \cap B = \{b, d\}$.
The elements b and d are the only elements that are in both sets A and B.



Example: Let $G = \{t, a, n\}$ and $H = \{n, a, t\}$. Then $G \cap H = \{a, n, t\}$.
Note that here $G = H = G \cap H$.

Example: Let $C = \{2, 6, 10, 14, \ldots\}$ and $D = \{2, 4, 6, 8, \ldots\}$.
Then $C \cap D = \{2, 6, 10, 14, \ldots\} = C$.

Example: Why is the location where a street and an avenue cross called an *intersection*? The location is contained in both the street and the avenue.

Example: Let $E = \{d, a, y\}$ and $F = \{n, i, g, h, t\}$. Then $E \cap F = \emptyset$.

Note: In all the examples, the intersection is a subset of each set forming the intersection, i.e., $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Disjoint Sets: Two sets whose intersection is the empty set are called *disjoint sets*.

Example: Let $E = \{d, a, y\}$ and $F = \{n, i, g, h, t\}$.
Since $E \cap F = \emptyset$, the sets E and F are disjoint sets.

Set Operations – Complement and Relative Complement

How does the operation of subtraction used in the following problem relate to sets?

Sam had \$836 in a checking account and wrote a check for \$429. How much money did Sam have in the checking account after the check was written?

The checking account is a set of dollars where the whole number 836 represents the cardinality of the set and the check represents a subset of the checking account with a cardinality of 429 where

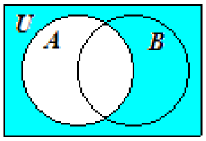
429 dollars are removed from the account. The subtraction, \$836 – \$429 = \$407, gives a new whole number that represents the cardinality of a set for the new checking account balance.

We develop subtraction of whole numbers through the set concept of removing the elements in a subset of a set from the set.

Remember that we often work with a specific set of objects when solving problems or discussing issues. We called this set of objects a *universal set* or *universe.* For example, in the lead-in problem above, the universal set could be either the set of all U. S. dollars or the set of the \$836 Sam originally had in the checking account.

Complement of a Set: The *complement* of a set, denoted A', is the set of all elements in the given universal set U that are not in A.

In set- builder notation, $A' = \{x \in U : x \notin A\}$.



The Venn diagram for the complement of set A is shown on the right where the shaded region represents A'.

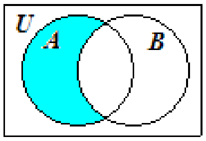
Example: For the lead-in example above, let the universal set U be the \$836 Sam originally has in the checking account and let A be the set of the \$429 of the check. The complement of set A would be the set of the \$407 remaining in the checking account.

Example: Let $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{1, 3, 5\}$. Then $A' = \{2, 4, 6\}$.

Example: $U' = \emptyset$ The complement of the universe is the empty set.

Example: $\emptyset' = U$ The complement of an empty set is the universal set.

Set Difference: The *relative complement* or *set difference* of sets A and B, denoted $A – B$, is the set of all elements in A that are not in B.



In set-builder notation, $A - B = \{x \in U : x \in A \text{ and } x \notin B\} = A \cap B'$.

The Venn diagram for the set difference of sets A and B is shown on the right where the shaded region represents $A – B$.

Example: For the lead-in example, let the universal set U be the set of all U.S. dollars, let set A be the set of \$836 Sam originally has in the checking account, and let B be the set of the \$429 of the check. Then the set difference of A and B would be the \$407 remaining in the checking account.

Example: Let $A = \{a, b, c, d\}$ and $B = \{b, d, e\}$. Then $A – B = \{a, c\}$ and $B – A = \{e\}$.

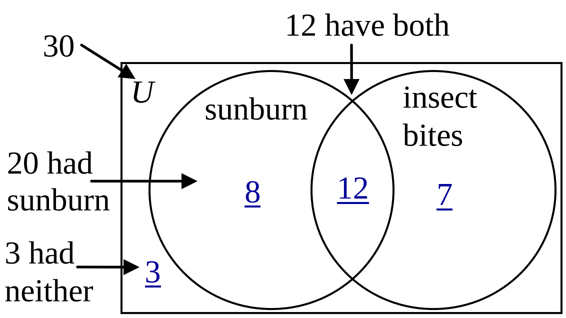
Example: Let $G = \{t, a, n\}$ and $H = \{n, a, t\}$. Then $G – H = \emptyset$.

Survey Problems

Problems similar to the problem posed at the beginning of this session can be represented using a Venn diagram, cardinal numbers, and the ideas of union and intersection. We use these concepts to solve the problem.

A group of 30 students went on a camping trip. Of these, 12 return with both sunburns and insect bites, and 20 report sunburn. How many suffered only insect bites if it is known that only three students suffered neither?

To solve it we construct a Venn diagram and enter the cardinal number for each section.



Solution: The 12 that had both are in the intersection of sunburn and insect bites.

The 3 that had neither are in the universe, but outside of the circles.

Since 20 had sunburn and 12 of those also had insect bites, we conclude 8 had only sunburn.

And finally, since there were 30 students total, and we have $3 + 8 + 12 = 23$ accounted for, there must be $30 – 23 = 7$ who had insect bites only.

If we let S represent the set of students who had a sunburn and B represent the set of students who suffered insect bites, we would have the following symbolic relationships with their interpretation.

Symbolic	Interpretation
$n(U) = 30$	Thirty students went on the camping trip.
$n(S \cup B) = 27$	Twenty-seven students received a sunburn or an insect bite.
$n(S \cap B) = 12$	Twelve students received a sunburn and an insect bite.
$n(S - B) = 8$	Eight students had only a sunburn.

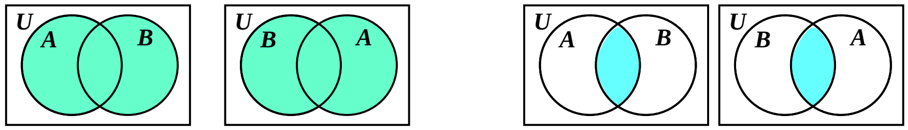
Properties of Set Operations

The following set properties are given here in preparation for the properties for addition and multiplication in arithmetic. Note the close similarity between these properties and their corresponding properties for addition and multiplication.

Commutative Properties: The *Commutative Property for Union* and the *Commutative Property for Intersection* say that the order of the sets in which we do the operation does not change the result.

General Properties: $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

Note that the shaded regions are the same for both sets of diagrams.



Example: Let $A = \{x : x$ is a whole number between 4 and 8$\}$ and $B = \{x : x$ is an even natural number less than 10$\}$. Then

$A \cup B = \{5, 6, 7\} \cup \{2, 4, 6, 8\} = \{2, 4, 5, 6, 7, 8\} = \{2, 4, 6, 8\} \cup \{5, 6, 7\} = B \cup A$

and $A \cap B = \{5, 6, 7\} \cap \{2, 4, 6, 8\} = \{6\} = \{2, 4, 6, 8\} \cap \{5, 6, 7\} = B \cap A$.

Associative Properties: The *Associative Property for Union* and the *Associative Property for Intersection* says that how the sets are grouped does not change the result.

General Property: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

Example: Let $A = \{a, n, t\}$, $B = \{t, a, p\}$, and $C = \{s, a, p\}$. Then

$(A \cup B) \cup C = \{p, a, n, t\} \cup \{s, a, p\} = \{p, a, n, t, s\} = \{a, n, t\} \cup \{t, a, p, s\} = A \cup (B \cup C)$

$(A \cap B) \cap C = \{a, t\} \cap \{s, a, p\} = \{a\} = \{a, n, t\} \cap \{a, p\} = A \cap (B \cap C)$.

Identity Property for Union: The *Identity Property for Union* says that the union of a set and the empty set is the set, i.e., union of a set with the empty set includes all the members of the set.

General Property: $A \cup \emptyset = \emptyset \cup A = A$

Example: Let $A = \{3, 7, 11\}$ and $B = \{x : x$ is a natural number less than 0$\}$.

Then $A \cup B = \{3, 7, 11\} \cup \{\} = \{3, 7, 11\}$.

The empty set is the identity element for the union of sets. What would be the identity element for the addition of whole numbers? What would be the identity element for multiplication of whole numbers?

Intersection Property of the Empty Set: The *Intersection Property of the Empty Set* says that any set intersected with the empty set gives the empty set.

General Property: $A \cap \emptyset = \emptyset \cap A = \emptyset$.

Example: Let $A = \{3, 7, 11\}$ and $B = \{x : x$ is a natural number less than 0$\}$.

Then $A \cap B = \{3, 7, 11\} \cap \{\} = \{\}$.

What number has a similar property when multiplying whole numbers? What is the corresponding property for multiplication of whole numbers?

Distributive Properties: The *Distributive Property of Union over Intersection* and the *Distributive Property of Intersection over Union* show two ways of finding results for certain problems mixing the set operations of union and intersection.

General Property: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example: Let $A = \{a, n, t\}$, $B = \{t, a, p\}$, and $C = \{s, a, p\}$. Then
$A \cup (B \cap C) = \{a, n, t\} \cup \{a, p\} = \{p, a, n, t\} = \{p, a, n, t\} \cap \{p, a, n, t, s\} = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = \{a, n, t\} \cap \{t, a, p, s\} = \{a, t\} = \{a, t\} \cup \{a\} = (A \cap B) \cup (A \cap C)$

More Terminology for Logic and Sets

How does the word "or" differ in the following two scenarios?

For condiments for your hotdog, you may have mustard or ketchup. What options do you have for your hotdog?

You are tired after a long week and are trying to decide whether to stay home or go out Friday evening. What options do you have for the evening?

For the first question, you may choose one or both of the condiments, that is, just mustard, or just ketchup, or both mustard and ketchup. For the second question, you cannot do both. You can only stay home or only go out.

The word "or" in the English language has two meanings depending on the context in which it is used. The first example with the hotdogs uses the *inclusive or*, which allows the choice of both options; whereas, the second example uses the *exclusive or*, which requires the choice of exactly one of the options.

Two different interpretations for the word "or" may cause confusion in finding solutions to problems in mathematics. To overcome this problem, mathematicians interpret the word "or" as an *inclusive or.*

Definition of "or" in Mathematics: The phrase "*a* **or** *b*" means we have only *a*, or only *b*, or both *a* and *b*.

Example: The statement, "The bicycle is red or it is a 10-speed." means that the bicycle could be only red, it could be only a 10-speed, or it could be both red and a 10-speed. This is the inclusive use of "or".

When working with sets, the word "or" corresponds to the set operation *union.*

Example: $\{ x : x$ is a factor of 12 *or* x is a factor of 10$\}$
$= \{ x : x \text{ is a factor of } 12\} \cup \{ x : x \text{ is a factor of } 10\}$
$= \{1, 2, 3, 4, 6, 12\} \cup \{1, 2, 5, 10\}$
$= \{1, 2, 3 ,4, 5, 6, 10, 12\}$

Unlike the word "or", the word "and" usually is not interpreted more than one way. For example, when we make a statement such as "Pat wants mustard and ketchup on the hotdog.", we understand that Pat wants both condiments. But, to make sure the interpretation is clear, we define the meaning of "and" for mathematics.

Definition of "and" in Mathematics: The phrase "*a* ***and*** *b*" means "both *a* and *b*."

Example: The statement, "You must make less than $10,000 a year and live in Minnesota to qualify for this grant.", means that you only qualify for the grant if you satisfy both conditions: you make less than $10,000 a year AND ALSO you live in Minnesota.

When working with sets, the word "and" corresponds to the set operation *intersection.*

Example: $\{ x : x \text{ is a factor of 12 } \textit{and } x \text{ is a factor of 10}\}$

$$= \{ x : x \text{ is an factor of 12}\} \cap \{ x : x \text{ is a factor of 10}\}$$

$$= \{1, 2, 3, 4, 6, 12\} \cap \{1, 2, 5, 10\}$$

$$= \{1, 2\}$$