Session 6 – Place Value

You contracted for a job to be paid and were to be paid $9,100 when the job was completed. The person wrote you a check made out for $1,900. Would you overlook the mistake since the person's only mistake was to transpose two digits? If not, why would you want the person to write a different check?

You would probably not accept the check because the interchanging of the two digits would be a huge change in the amount of money paid. In our numeral system (the Hindu-Arabic numeration system), the position of the numerals is important. The Hindu-Arabic numeration system is a *place-value* system, which means that the position of the numerals affects the value of the number it represents. (Remember that a numeral is the symbol and that a number is the value a numeral represents.) In the situation above, the 9 in the $9,100 represents nine thousand dollars and the 1 represents one hundred dollars. In $1,900, the 9 represents nine hundred dollars and the 1 represents one thousand dollars.

Notice the close relationship the place-value system has to sets, cardinality, and the set operation of union. The amount of money represented by $9,100 may be thought of as the cardinality of the union of a set containing nine thousand dollars with a set containing one hundred dollars. Similarly, the amount of money represented by $1,900 may be thought of as the cardinality of the union of a set containing one thousand dollars with a set containing nine hundred dollars.

The Hindu's developed the system before the 9th century. The Persian mathematician al-Khwarizmi wrote a book on the system in about 825 A.D. after which the system was adopted by the Arabs. Though the system was first introduced into Europe in the 10th century, it was not widely used in Europe until after the invention of the printing press in the 15th century.

In this session, we will consider the structure of the Hindu-Arabic numeration system and how it relates to the standard algorithm for addition.

Hindu-Arabic Numeration System

The Hindu-Arabic numeration system is a base ten place value system that uses ten numerals. The ten numerals (or digits) are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Only a single digit may be placed in each place value position. The place value of each column is ten times greater in value than the place value in the column to its right.

×10 ×10 ×10 ×10 ×10 ×10 ×10 ×10

100,000,000	10,000,000	1,000,000	100,000	10,000	1,000	100	10	1
hundred million$	ten million$	million$						

The bottom part of the above table illustrates the relationship the place-value has to the cardinality of sets of objects: set of $1, set of $10, set of $100, …, set of one hundred million dollars. Since it is difficult to illustrate large sets of one-dollar bills in a compact form, the illustration uses pictures of bills for $1, $10, $100, $1000, $10,000 and $100,000. (Note that the U.S. Treasury no longer makes bills above the $100-bill and they are no longer in circulation.)

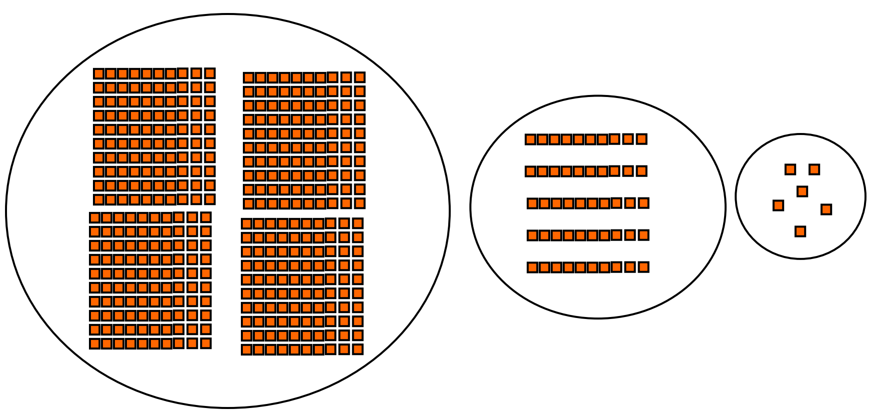
To write "six thousand" using the Hindu-Arabic numeration system (base ten place value system), the numeral 6 is placed in the thousands' column and zeros are used to fill in the positions to the right, so that the 6 remains in the 4th position from the right when the place value columns are not included.

	100,000,000	10,000,000	1,000,000	100,000	10,000	1,000	100	10	1	
six thousand =						6	0	0	0	= 6,000
five million =			5	0	0	0	0	0	0	= 5,000,000
thirty million five hundred =		3	0	0	0	0	5	0	0	= 30,000,500

Expanded Notation

One way to show the place values of a numeral without writing out the entire table is to use a form of expanded notation. In the following form of expanded notation, we rewrite a numeral by separating the numeral into its place values and adding those values. This is the same as breaking a set into the union of several disjoint sets.

Example: 456 = 400 + 50 + 6



Example: 800,201 = 800,000 + 0 + 0 + 200 + 0 + 1

Example: 13<u>0</u>,54<u>0</u> = 100,000 + 30,000 + <u>0</u> + 500 + 40 + <u>0</u>

		no thousands			no ones

Notice that in this form, each addend has at most one non-zero digit. For positions that have the numeral 0, we keep the 0 in the expanded notation to show that there is no place value for that position.

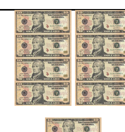
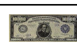
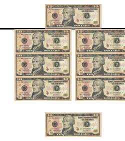
The Standard Algorithm for Addition

Many of the procedures we use to perform arithmetic with whole numbers follow directly from the nature of our place value system (Hindu-Arabic numeration system).

For instance, when we use the Standard Algorithm for Addition, why do we line up the addition of whole numbers so that the numerals are "right justified"? The reason follows from the concepts of the union of disjoint sets and place value.

Example: 456,090 + 4,593 + 12,079 + 300

We illustrate the problem by using sets and denominations of money.

	hundred thousands	ten thousands	thousands	hundreds	tens	ones
456,090						
4,593						
12,079						
+ 300						

We use the above example to illustrate how the place value system and the union of sets demonstrates the step-by-step procedure of the Standard Addition Algorithm.

$$\begin{array}{r} 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline \end{array}$$

We begin adding the column farthest to the right and then when the total exceeds 9, we exchange each set of ten 1's for a ten and place it in the next column. The concept is the same as when we combine the sets of $1-bills and, when possible, convert them to $10-bills. The whole procedure is the grouping of 10's and placing them in the correct place value. Different people give different names for this procedure; some call it "regrouping", "exchanging", "trading", or "carrying".

We add the ones column to obtain 12, e.g., 0 + 3 + 9 + 0 = 12. That is, we have combined the dollar bills to obtain twelve dollars and then exchanged ten $1-bills for one $10-bill. The 2 for the remaining two dollars is written at the bottom of the column and the 1 for the $10-bill is written at the top of the tens column.

$$\begin{array}{r} 1 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 2 \end{array}$$

$$\begin{array}{r} 21 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 62 \end{array}$$

We next add the ten's column, 1 + 9 + 9 + 7 + 0 = 26. We have combined the $10-bills to obtain 26 $10-bills and then exchanged twenty of them for two $100-bills. The 6 for the remaining $10-bills is written at the bottom of the column and the 2 for the $100-bills is written at the top of the hundreds column.

We next add the hundred's column, 2 + 0 + 5 + 0 + 3 = 10. We have combined the $100-bills to obtain 10 $100-bills and then exchanged them for one $1000-bill. Since there are no $100-bills remaining, we record a 0 at the bottom of the column. We record a 1 at the top of the thousands column for the one $1000-bill.

$$\begin{array}{r} 1\ 21 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 062 \end{array}$$

$$\begin{array}{r} 1\ 1\ 21 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 3{,}062 \end{array}$$

We next add the thousand's column, 1 + 6 + 4 + 2 = 13. We have combined the $1000-bills to obtain 13 $1000-bills. We exchange ten of them for one $10,000-bill. The 3 for the remaining $1000-bills is written at the bottom of the column and the 1 for the $10,000-bill is written at the top of the ten-thousands column.

We next add the ten-thousand's column, 1 + 5 + 1 = 7. We have combined the $10,000-bills to obtain 7 $10,000-bills. Since we have less than ten of them, we are unable to make any exchanges. So, we record the 7 at the bottom of the column.

$$\begin{array}{r} 1\ 1\ 21 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 73{,}062 \end{array}$$

$$\begin{array}{r} 1\ 1\ 21 \\ 456{,}090 \\ 4{,}593 \\ 12{,}079 \\ +\quad 300 \\ \hline 473{,}062 \end{array}$$

Finally, since there is only a 4 in the hundred-thousand's column, we write it at the bottom of the column. Since there were only 4 $100,000-bills, we had none to combine. So, we keep those bills.

Hence, 456,090 + 4,593 + 12,079 + 300 = 473,062.