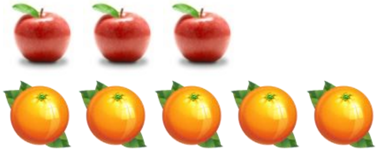
Session 7 – Order and Addition of Whole Numbers

How do sets relate to the comparison being made in the following problem?

Sam has three apples and Terry has five oranges. Who has more pieces of fruit?



Terry has more fruit. We illustrate with the set concepts of 1-1 correspondence and subset. The three apples can only be set up with a 1-1 correspondence with three of the oranges. Two of the oranges cannot be paired with any apples in the 1-1 correspondence. That is, the apples can only be paired with a proper subset of the set of oranges. Since the set of apples is equivalent to a proper subset of the set of oranges, Sam has fewer apples than Terry has oranges.

The relationship above motivates the definitions for ordering the natural and whole numbers that we give in this session. We begin with a quick reminder of the definitions of the sets of natural numbers and whole numbers.

N = Natural numbers = {1, 2, 3, 4, 5, 6, . . . }. Also, called "counting numbers".

W = Whole numbers = {0, 1, 2, 3, 4, . . . }.

Order

Order: The *order* of whole numbers is a comparison of the relative size of sets (comparing cardinal numbers). For sets A and B with cardinal numbers $a = n(A)$ and $b = n(B)$, if A is equivalent to a proper subset of B, then we say that a is less than b or b is greater than a.

Notation: $a < b$ or $b > a$, these are read as *a is less than b* and *b is greater than a*.

Example: Pat sold several items on eBay: \$25, \$36, \$12, \$21, \$34, \$22, \$15, and \$19. Order the values in this list from least to greatest.

Solution: \$12, \$15, \$19, \$21, \$22, \$25, \$34, \$36

We illustrate the above solution below where we use sets of dollar signs.

12	\$\$\$\$\$\$\$\$\$\$\$\$	Note that each set of dollars can be set up with a 1-1 correspondence with a *proper subset* of the set below it.
15	\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$	
19	\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$	
21	\$	
22	\$	
25	\$	
34	\$	
36	\$	

$$12 < 15 < 19 < 21 < 22 < 25 < 34 < 36$$

We need to pay attention to whether we are asked to order the list from least to greatest or greatest to least.

What do we mean by ≤ (less than or equal to) and ≥ (greater than or equal to)? We modify the definition for order for these cases. In this more general form for ordering whole numbers, we use subset in the definition in place of proper subset since subsets allow sets to be equivalent.

Order (A more general definition): For sets A and B with cardinal numbers $a = n(A)$ and $b = n(B)$, if A is equivalent to a subset of B, then we say that a is less than or equal to b or b is greater than or equal to a.

Notation: $a \leq b$ or $b \geq a$, these are read as *a is less than or equal to b* and *b is greater than or equal to a*.

Example: Pat sold several items on eBay: \$22, \$34, \$12, \$22, \$34, \$22, \$12, and \$19. Order the values in this list from least to greatest.

Solution: \$12, \$12, \$19, \$22, \$22, \$22, \$34, \$34

We illustrate the above solution below where we use sets of dollar signs.

12	\$\$\$\$\$\$\$\$\$\$\$\$	Note that each set of dollars can be set up with a 1-1 correspondence with a *subset* of the set below it. Also, note that some of the sets are equivalent.
12	\$\$\$\$\$\$\$\$\$\$\$\$	
19	\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$\$	
22	\$	
22	\$	
22	\$	
34	\$	
34	\$	

Written with the symbols: $12 \leq 12 \leq 19 \leq 22 \leq 22 \leq 22 \leq 34 \leq 34$.

Also, note that $12 = 12 < 19 < 22 = 22 = 22 < 34 = 34$

More on Inequality Symbols

The symbol $<$ means “is strictly less than” since we are comparing the cardinal numbers for two sets where one set is equivalent to a proper subset of the other set.

Example: If $A \subset B$, then $n(A) < n(B)$.

Example: If Abby has \$4 and Billy has \$5, then Abby has less money than Billy.

Symbolically: If $n(A) = 4$ and $n(B) = 5$, then $4 < 5$.
The 4 is strictly less than the 5 and set A is equivalent to a proper subset of set B.

The symbol $>$ means “is strictly greater than” since we are comparing the cardinal numbers for two sets where one set is equivalent to a proper subset of the other set.

Example: If $A \subset B$ and B is a finite set, then $n(B) > n(A)$.

Example: If Ann has five apples and Bob has four bananas, then Ann has more apples than Bob has bananas.

Symbolically: If $n(A) = 5$ and $n(B) = 4$, then $5 > 4$.
The 5 is strictly greater than 4 and the set B is equivalent to a proper subset of set A.

The symbol $\leq$ means "is less than or equal to" since we are comparing the cardinal numbers for two sets where one set is equivalent to a subset of the other set. Remember that if a set is a subset of another set, the two sets may be the same set.

Example: If $A \subseteq B$, then $n(A) \leq n(B)$.

Example: The most goals the hockey team scored in a game this year was seven. This means that the number of goals scored in each game was less than or equal to seven.

Symbolically: If $n(G) = g$ where G is the set of individual goals scored in a game this year, then $g \leq 7$.
The set G is equivalent to a subset of set L where L is the goals scored in a game with seven goals. Note that either $g < 7$ or $g = 7$.

Example: If $n(A) = 6$ and $n(B) = 6$, then $6 \leq 6$.
In this case, the two sets are equivalent. Note that in this case, we also have $6 = 6$.

The symbol $\geq$ means "is greater than or equal to" since we are comparing the cardinal numbers for two sets where one set is equivalent to a subset of the other set.

Example: If $A \subseteq B$, then $n(B) \geq n(A)$.

Example: We hope to make a profit of at least \$35 when we sell the table. This means that the amount of profit should be greater than or equal to \$35.

Symbolically: If P is the set of dollars of profit, then $n(P) \geq 35$.
A set containing 35 dollars is equivalent to a subset of set P. Note that either $n(P) > 35$ or $n(P) = 35$.

Compound Inequalities

Each week for the past year, Raul's Balloon Emporium's weekly profit was as high as \$9,540 and as low as \$4,274.

We often express problems of this type algebraically with a compound inequality. We may express the problem as

$$4{,}274 \leq P \leq 9{,}540$$

where P represents a weekly profit. The expression may be interpreted as "all the values P for which P is greater than or equal to 4,274 and P is less than or equal to 9,540.

Based on what we did earlier in this session, what would the interpretation be in terms of the language of sets? Or, how does this problem relate to sets?

Example: $4 < x \leq 10$ means
"all the values of x for which 4 is less than x, and x is less than or equal to 10."

Important Note: Do ***not*** mix greater than and less than symbols in the same compound inequality. The inequality signs in a compound inequality should both point the same direction.

Compound inequalities are often written in set-builder notation. Notice how the “universal set” is specified in these examples.

Example: $\{x : 4 < x \leq 10,\ x \in N\} = \{5, 6, 7, 8, 9, 10\}$

Example: $\{x : 0 \leq x < 5,\ x \in W\} = \{0, 1, 2, 3, 4\}$

Union of Sets and Addition

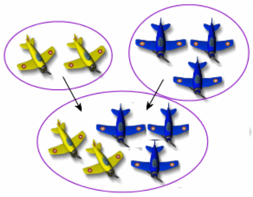
How does the operation of addition used in the following problem relate to sets?

> *Sam had two savings accounts. He had $836 in one account and $429 in another account. How much money did Sam have in savings?*

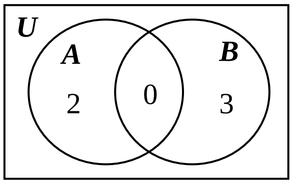
The first account is a set of dollars where the whole number 836 represents the cardinality of the set and the second account is a set of dollars with a cardinality of 429. The two accounts are distinct; hence, the two sets do not have any elements (dollars) in common. The addition, $836 + $429 = $1,265, gives a new whole number that represents the cardinality of a new set which is the union of the two disjoint sets (accounts).

In this section, we illustrate the relationships between the addition of whole numbers and the union of sets. We show a difference in the process of addition when the two sets are disjoint and when their intersection is not the empty set.

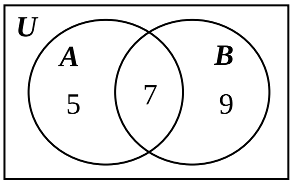
Andy has two airplanes and Billy has three airplanes. How many airplanes do they have all together?



In the picture and the Venn diagram, there are two sets *A* and *B*. Notice that the sets are disjoint,
$n(A) = 2$, $n(B) = 3$, and $n(A \cup B) = 2 + 3 = 5$.



Ann has $12 available to spend and Bob has $16 available to spend. But, $7 of their money is in a joint account. How much money do they have all together?



In this Venn diagram, there are two sets *A* and *B*.
Notice that the sets are ***not*** disjoint, $n(A) = 12$, $n(B) = 16$,
but now $(A \cup B) = 21 \neq 12 + 16 = 28$.

Can you explain why you cannot just add 12 and 16 to find the total?

Notice that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ and this formula works for both of the Venn diagram situations above. We will see this relationship again when we study probability.

The above problems involving sets motivate the following definition for addition of whole numbers.

Addition of Whole Numbers: Let $a = n(A)$ and $b = n(B)$ where A and B are two disjoint finite sets. Then

$$a + b = n(A \cup B).$$

The whole numbers a and b are called *addends*
and the result $a + b$ is called the *sum*.
(Reminder: n(A) represents the cardinal number for set A.)

$$\overbrace{3 + 5}^{\textit{addends}} = \underset{\uparrow \atop \textit{sum}}{8} \qquad \overbrace{a + b}^{\textit{addends}} = \underset{\uparrow \atop \textit{sum}}{c}$$

Important Note. The sets A and B must be disjoint sets, i.e., $A \cap B = \emptyset$.

Some Shorthand Tips for Addition of Whole Numbers

1. Note that 9 = 10 – 1, that is, adding 9 is like "adding 10 and 1 less."

   Examples:

   9 + 7 *is like*
   10 + 6 = 16

   23 + 9 *is like*
   22 + 10 = 32

   138 + 9 *is like*
   137 + 10 = 147

2. Use the associative and commutative properties for addition of whole numbers (these properties will be discussed in the next session). The properties allow us to add whole numbers in any order. When adding, look for pairs whose sum is 10.

   Examples:

   $$\begin{array}{rr} & \cancel{4} \\ & 8 \\ + & \cancel{6} \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{rr} & 10 \\ + & 8 \\ \hline & 18 \end{array}$$

   $$3 + \cancel{9} + \cancel{5} + \cancel{1} + \cancel{5}$$

   becomes

   10 + 10 + 3 = 23

3. Use expanded notation to "see" what happens when you are adding down the columns in the standard algorithm.

   Examples:

   $$\begin{array}{rrcrcr} & 49 & = & 40 & + & 9 \\ & 26 & = & 20 & + & 6 \\ & 71 & = & 70 & + & 1 \\ & 64 & = & 60 & + & 4 \\ + & \underline{38} & = & \underline{30} & + & \underline{8} \\ & & & 220 & + & 28 = \boxed{248} \end{array}$$

   $$\begin{array}{rrcrcrcr} & 423 & = & 400 & + & 20 & + & 3 \\ & 262 & = & 200 & + & 60 & + & 2 \\ + & \underline{135} & = & \underline{100} & + & \underline{30} & + & \underline{5} \\ & & = & 700 & + & 110 & + & 10 = \boxed{820} \end{array}$$