Session 8 – Addition and Subtraction

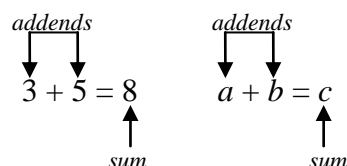## *Review Addition Vocabulary*

Addition of Whole Numbers: Let $a = n(A)$ and $b = n(B)$ where A and B are two disjoint finite sets. Then

$$a + b = n(A \cup B).$$

Addends are the values that are being added.

The **sum** is the answer to an addition problem.

$$\underset{sum}{\underset{\uparrow}{3 + 5 = \underset{\uparrow}{8}}} \qquad \underset{sum}{\underset{\uparrow}{a + b = \underset{\uparrow}{c}}}$$

addends   *addends*

Properties for Addition of Whole Numbers

Commutative Property of Addition: The *Commutative Property for Addition of Whole Numbers* says that the order of the addends does not change the sum.

General Property: $a + b = b + a$

Relationship to sets:
   Use the set definition for addition and the Commutative Property for the Union of Sets:
   Let A and B be two disjoint finite sets with cardinal numbers $n(A)$ and $n(B)$. Then
   $$n(A) + n(B) = n(A \cup B) = n(B \cup A) = n(B) + n(A).$$

Numeric Example: $3 + 5 = 8 = 5 + 3$

Algebraic Example: $3x + 4x = 7x = 4x + 3x$

Associative Property of Addition: The *Associative Property for Addition of Whole Numbers* says that how the addends are grouped does not change the sum.

General Property: $(a + b) + c = a + (b + c)$

Relationship to sets:
   Use the set definition for addition and the Associative Property for the Union of Sets:
   Let A, B, and C be three disjoint finite sets with cardinal numbers $n(A)$, $n(B)$, and $n(C)$. Then
   $$\begin{aligned} [n(A) + n(B)] + n(C) &= n([A \cup B] \cup C) \\ &= n(A \cup [B \cup C]) \\ &= n(A) + [n(B) + n(C)]. \end{aligned}$$

Numeric Example: $\begin{aligned}(2 + 5) + 8 &= 7 + 8 \\ &= 15 \\ &= 2 + 13 \\ &= 2 + (5 + 8)\end{aligned}$

Algebraic Example: $(2x + 3x) + 4x = 5x + 4x$
$$= 9x$$
$$= 2x + 7x$$
$$= 2x + (3x + 4x)$$

Identity Property of Addition: The *Identity Property for Addition of Whole Numbers* says that the sum of a number and zero is the number. Zero is called the *additive identity*.

General Property: $a + 0 = 0 + a = a$

Relationship to sets:
Use the set definition for addition and the Identity Property for the Union of Sets:
Let *A* be a finite set with cardinal number *n*(*A*). Then
$$n(A) + n(\emptyset) = n(A \cup \emptyset) = n(A).$$

Numeric Example: $5 + 0 = 5$ and $0 + 5 = 5$

Algebraic Example: $3x + 0 = 3x$ and $0 + 3x = 3x$

Introduction to Subtraction of Whole Numbers

Subtraction is often considered to be the inverse operation of addition, that is,
$$a - b = c \text{ only when } b + c = a.$$
We may use this inverse relationship to check the answers to subtraction problems. In this session, we also learn how to use this inverse relationship to solve addition and subtraction equations. But, first, we consider how subtraction relates to sets and use this relationship to motivate subtraction as an inverse operation for addition.

How does the operation of subtraction used in the following problem relate to sets?

Sam had $836 in a checking account and wrote a check for $429. How much money did Sam have in the checking account after the check was written?

The checking account is a set of dollars where the whole number 836 represents the cardinality of the set and the check represents a subset of the checking account with a cardinality of 429 where 429 dollars are removed from the account. The subtraction, $836 – $429 = $407, gives a new whole number that represents the cardinality of a set for the new checking account balance.
We develop subtraction of whole numbers through the set concept of removing the elements in a subset of a set from the set.

Subtraction of Whole Numbers

How should we define the subtraction of whole numbers?

In the lead-in example, the remaining balance was the difference between the cardinalities of the sets for the checking account and the check. This also works for the third example where $n(G) - n(H) = 3 - 3 = 0 = n(\emptyset)$. But, with the second example the difference between the cardinalities does not give the expected result, e.g., $n(A) - n(B) = 4 - 3 = 1 \neq 2 = n(A - B)$. In this case, B is not a subset of A. This leads to the following set definition for subtraction of whole numbers.

Set Definition for Subtraction of Whole Numbers: Let $a = n(A)$ and $b = n(B)$ where sets A and B are two finite sets such that $B \subseteq A$. Then $a - b = n(A - B)$. The result $a - b$ is called the *difference*. The a is called the *minuend* and the b is called the *subtrahend*.

Investigation. Does the above definition apply to each of the following problems? If not, how should we modify the definition for subtraction?

1. A parent had \$47 and gave \$25 to a child. How much cash did the parent have left?

2. Lynn had to buy twelve books for this semester and Pat had to buy nine books for this semester. How many more books did Lynn have to buy than Pat?

3. Sam has 483 songs on a MP3 player. How many more songs does Sam need to have 1000 songs?

The first problem fits the definition since the \$25 given to the child is a subset of the \$47 the parent originally had. The parent has \$22 left since $47 - 25 = 22$.

The second problem does not fit the above definition since Pat's books are not a subset of Lynn's books. We would still consider the problem to be $12 - 9 = 3$, i.e., Lynn has three more books than Pat. But, Pat's books can be put into a one-to-one correspondence with a subset of Lynn's books, so with this modification the problem can be made to fit a modified version of the above definition.

The third problem fits the definition where songs Sam currently has on the MP3 player is a subset of the songs Sam desires to have. Sam would like 517 more songs since $1000 - 483 = 517$. Note that this problem could be thought of as an addition problem, $483 + N = 1000$, where N represents the number of additional songs Sam would like to have. This motivates the definition of subtraction of whole numbers to be the inverse of addition that we mentioned at the beginning of this section.

Inverse Definition for Subtraction of Whole Numbers: Let a and b be whole numbers. Then $a - b = c$ if and only if there is a whole number c such that $b + c = a$. The a is called the *minuend* and the b is called the *subtrahend*.

Standard Subtraction Algorithm

We motivate the standard subtraction algorithm with both the set model and expanded notation.

Example: Consider the problem 40 – 7.

Standard	Expanded Notation	Set Model
$\begin{array}{r} 40 \\ -\,7 \\ \hline \end{array}$	$\begin{array}{rr} 40 & +\quad 0 \\ & -\quad 7 \\ \hline \end{array}$	

We are unable to remove \$7 from \$40 without exchanging a \$10-bill because there are no \$1-bills to remove. In the symbolic form, we cannot take 7 ones from 0 ones because we are unable to remove elements from an empty set.

$\begin{array}{r} {\scriptstyle 3\;\;10} \\ \cancel{4}\;\cancel{0} \\ -\,7 \\ \hline \end{array}$	$\begin{array}{rr} 30 & 10 \\ \cancel{40} & +\quad \cancel{0} \\ & -\quad 7 \\ \hline \end{array}$	

We exchange a \$10-bill for 10 \$1-bills. In symbolic form, we exchange one ten for ten ones.

$\begin{array}{r} {\scriptstyle 3\;\;10} \\ \cancel{4}\;\cancel{0} \\ -\,7 \\ \hline 3\;\;3 \end{array}$	$\begin{array}{rr} 30 & 10 \\ \cancel{40} & +\quad \cancel{0} \\ & -\quad 7 \\ \hline 30 & +\quad 3 \end{array}$	

We remove 7 \$1-bills. In symbolic form, we subtract 7 ones from ten ones.

Example: Consider the problem 350 – 125.

Standard	Expanded Notation	Set Model
$\begin{array}{r} 350 \\ -\,125 \\ \hline \end{array}$	$\begin{array}{rrr} 300 & +\;\; 50 & +\;\; 0 \\ -\;\;100 & -\;\; 20 & -\;\; 5 \\ \hline \end{array}$	

We are unable to remove \$5 from \$350 without exchanging a \$10-bill because there are no \$1-bills to remove. In the symbolic form, we cannot take 5 ones from 0 ones because we are unable to remove elements from an empty set.

$\begin{array}{r} {\scriptstyle 4\;\;10} \\ 3\;\cancel{5}\;\cancel{0} \\ -\,1\;2\;5 \\ \hline \end{array}$	$\begin{array}{rrr} & 40 & 10 \\ 300 & +\;\; \cancel{50} & +\;\; \cancel{0} \\ -\;\;100 & -\;\; 20 & -\;\; 5 \\ \hline \end{array}$	

We exchange a \$10-bill for 10 \$1-bills. In symbolic form, we exchange one ten for ten ones.

$\begin{array}{r} 410 \\ 3\;\cancel{5}\;\cancel{0} \\ -\;1\;2\;5 \\ \hline 2\;2\;5 \end{array}$	$\begin{array}{r} 40\;10 \\ 300\;+\;\cancel{50}\;+\;\cancel{0} \\ -\;100\;-\;20\;-\;5 \\ \hline 200\;+\;20\;+\;5 \end{array}$

We remove 5 \$1-bills, 2 \$10-bills, and 1 \$100-bill. In symbolic form, we subtract five ones from ten ones, two tens from four tens, and one hundred from three hundreds.

Example: Consider the problem 500 – 375.

Standard	Expanded Notation	Set Model
$\begin{array}{r} 500 \\ -\;375 \end{array}$	$\begin{array}{r} 500\;+\;00\;+\;0 \\ -\;300\;-\;70\;-\;5 \end{array}$	

Regrouping is necessary in this problem in both the ones' and tens' columns.

$\begin{array}{r} 9 \\ 4\;\cancel{10}\;10 \\ \cancel{5}\;\cancel{0}\;\cancel{0} \\ -\;3\;7\;5 \end{array}$	$\begin{array}{r} 90 \\ 400\;\cancel{100}\;10 \\ \cancel{500}\;+\;\cancel{00}\;+\;\cancel{0} \\ -\;300\;-\;70\;-\;5 \end{array}$	

We exchange a \$100-bill for 10 \$10-bills and then exchange 1 \$10-bill for 10 \$1-bills. In symbolic form, we exchange one hundred for ten tens, and then exchange one ten for ten ones.

$\begin{array}{r} 9 \\ 4\;\cancel{10}\;10 \\ \cancel{5}\;\cancel{0}\;\cancel{0} \\ -\;3\;7\;5 \\ \hline 1\;2\;5 \end{array}$	$\begin{array}{r} 90 \\ 400\;\cancel{100}\;10 \\ \cancel{500}\;+\;\cancel{00}\;+\;\cancel{0} \\ -\;300\;-\;70\;-\;5 \\ \hline 100\;20\;5 \end{array}$	

We remove 5 \$1-bills, 7 \$10-bills, and 3 \$100-bills. In symbolic form, we subtract 5 ones from ten ones, 7 tens from 9 tens, and 3 hundreds from 4 hundreds.

Try this subtraction problem by first showing the expanded notation and then the short cut form. Be sure you understand the reasoning behind the regroupings.   Problem: 802 – 548.

Solving Equations with Properties of Equality

Solution to an Equation: A value is a solution to an equation when that value is substituted for the variable in the equation, the resulting arithmetic statement is a true statement.

> Example: Is 3 a solution to the equation $4 + x = 7$?
> Since $4 + 3 = 7$, it is true that 3 is a solution to the equation.
> We state that 3 is a solution to this equation by writing $x = 3$.
> We write the solution set to the equation as $\{3\}$.

> Example: The value 4 is *not* a solution to the equation $10 - x = 5$ since $10 - 4 \neq 5$.

> Example: The value 30 is a solution to $x - 12 = 18$ since $30 - 12 = 18$.
> The solution set for the equation is $\{30\}$.

Addition Property of Equality: If we add the same amount to both sides of an equation, the resulting statement is still an equation and it has the same solution set as the original equation.

> General Property: If $a = b$, then $a + c = b + c$.

> Example: Solve $x - 23 = 70$. We illustrate two methods.

> Method 1:
> $$x - 23 = 70$$
> $$(x - 23) + 23 = 70 + 23 \qquad \text{Addition Property of Equality}$$
> $$x = 93$$

> Method 2:
> $$\begin{array}{rcl} x \;-\; 23 &=& 70 \\ +\;23 & & +23 \quad \blacktriangleleft\text{---} \; 23 \text{ is added to both sides of the equation} \\ \hline x \;+\; 0 &=& 93 \\ \\ x &=& 93 \end{array}$$

> Check the solution by substitution: $93 - 23 = 70$.

Subtraction Property of Equality: If we subtract the same amount from both sides of an equation, the resulting statement is still an equation and it has the same solution set as the original equation.

> General Property: If $a = b$, then $a - c = b - c$.

> Example: Solve $x + 19 = 59$. We illustrate two methods.

> Method 1:
> $$x + 19 = 59$$
> $$(x + 19) - 19 = 59 - 19 \qquad \text{Subtraction Property of Equality}$$
> $$x = 40$$

Method 2:
$$\begin{array}{r} x \;+\; 19 \;=\; 59 \\ \underline{-19 \qquad -19} \\ x \;+\; 0 \;=\; 40 \end{array}$$ ◄ - - 19 is subtracted from both sides of the equation

$$x \quad\;=\; 40$$

Check the solution by substitution: $40 + 19 = 59$.