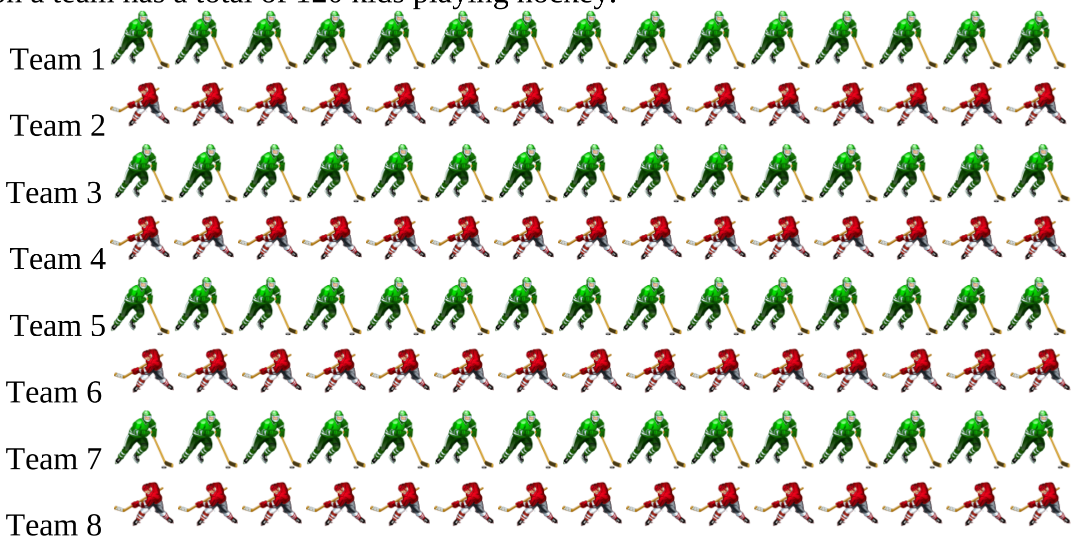
Session 9 – Models of Multiplication

How does the operation of multiplication used in the following problem relate to sets?

> *The Moorhead Youth Hockey League has eight teams in an age group with fifteen players on each team. How many kids play hockey in this age group?*

The league consists of a set of eight teams and each team is a set of fifteen players. We note that each player is on only one team. We consider all the pairings of a player with a specific team to be a set. The cardinality of this set of player-team pairings is 120. The league with eight teams of fifteen players on a team has a total of 120 kids playing hockey.



Also, note how the solution is the cardinality of the union of the eight disjoint sets (teams). In other words, we may consider the problem to be an addition problem where

$$8 \times 15 = 15 + 15 + 15 + 15 + 15 + 15 + 15 + 15 = 120.$$

This relationship motivates the following definition for multiplication of whole numbers.

Repeated Addition Definition for Multiplication of Whole Numbers. Given a whole number $a \neq 0$ of equal sets, each containing b elements, we define $0 \cdot b = 0$ and

$$ab = \underbrace{b + b + \cdots + b}_{a \text{ terms}}$$

The numbers a and b are called *factors* and ab is the *product.*

Notations used for multiplication: $ab = a \cdot b = a \times b = a * b = (a)(b)$.

Example: The product 5×42 may be thought of as $42 + 42 + 42 + 42 + 42$.
When we add these addends, the sum is 210, so $5 \times 42 = 210$.

Array Model for Multiplication

Another model for multiplication of whole numbers is the sets may be interpreted as an *array of rows and columns*. Usually in mathematics, rows are horizontal and columns are vertical.

Example: What is the area of a region that measures three feet by eight feet?
The product 3 × 8 may be illustrated as three rows of eight items,
as in this figure with 3 rows of 8 squares each:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24

We see that 3 rows of 8 squares result in 24 squares. So 3 × 8 = 24.
(Each square has area 1 square foot.)

This is repeated addition since each row may be considered to be a set, so the cardinality of the union of the three disjoint sets gives us 3 × 8 = 8 + 8 + 8 = 24.
The area of the region is twenty-four square feet.

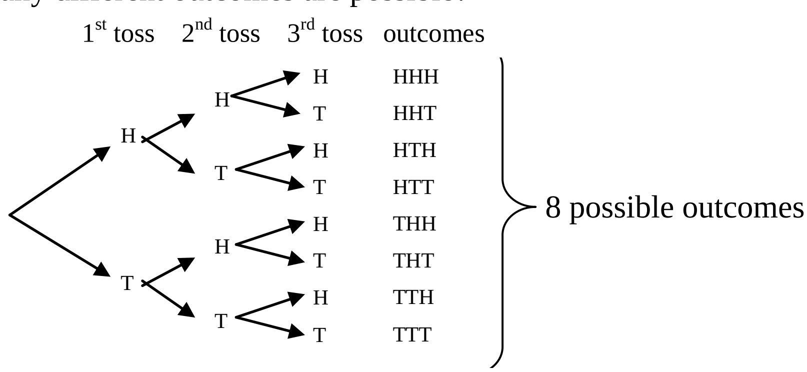
Example: How many children are in a room with fourteen children in each of three rows?

Since 3 × 14 = 14 + 14 + 14 = 42, there are 42 children in the room.

Tree Diagram

A tree diagram is another way to illustrate multiplication to figure out how many ways something can happen. It is useful in problems where several choices or stages follow one after another. Each choice or stage is represented by a branching. The total number of end-branches of the tree diagram shows us all the different ways the choices or stages can happen..

Example: A coin is tossed three times and H (heads) or T (tails) is recorded for each toss.
How many different outcomes are possible?



2 × 2 × 2 = 8 WAYS

Where three coins are tossed there are eight possible distinct outcomes.
Note that the order of the outcomes is important.

The above tree diagram does not seem to match the repeated addition concept for multiplication though it is possible to interpret it as repeated addition. How would you illustrate the problem as repeated addition? The problem does motivate another way of looking at multiplication. Since the order is important in the tree diagram, the set of solutions in the above problem is a set of ordered triplets; that is, HHT is not the same as HTH or THH even though all three have two heads and one tail. This motivates the next method which we call a Cartesian Product.

The Language of Sets— Cartesian Product

Consider the following array of ordered pairs of numbers where the first number is the row number and the second number in the pair is the column number. Note the shaded box is in the second row and fourth column represented with the ordered pair (2, 4).

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)

We note that the table has 3(5) = 15 small rectangular regions. We develop this concept in terms of a set operation that will be used to define multiplication.

Ordered Pair: An *ordered pair* is a pair of objects where one element is designated first and the other element is designated second, denoted (a, b).

Cartesian Product: The *Cartesian product* of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs where the elements of A are first and the elements of B are second.

In set-builder notation, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Example: Let $A = \{\text{H, T}\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

$A \times B$ = {(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),
(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)}

$B \times A$ = {(1, H), (2, H), (3, H), (4, H), (5, H), (6, H),
(1, T), (2, T), (3, T), (4, T), (5, T), (6, T)}

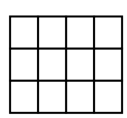
Note that in this case $A \times B \neq B \times A$, i.e., the Cartesian product is not commutative.
Also, note that $n(A) \cdot n(B) = 2(6) = 12 = n(A \times B)$.

Example: $A \times \emptyset = \emptyset$ since no ordered pairs can be formed when one of the sets is empty.
Also, note that $n(A) \cdot n(\emptyset) = 2(0) = 0 = n(A \times \emptyset)$.

Cartesian Product Definition for Multiplication of Whole Numbers. Let A and B be two finite sets with $a = n(A)$ and $b = n(B)$. Then $ab = n(A \times B)$.
The numbers a and b are called *factors* and ab is the *product.*

Two common methods for illustrating a Cartesian product are an array and a tree diagram.

Example: A small village has four streets and five avenues laid out in a rectangular grid. How many intersections are there?

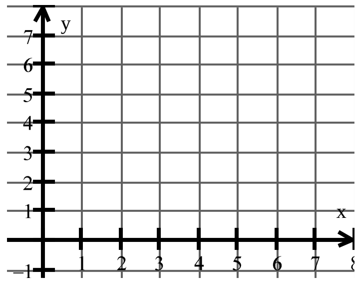


We have two sets, streets (S) and avenues (A). The elements from the two sets form a list of ordered pairs such as the intersection of 1^{st} Street and 2^{nd} Avenue, (1, 2). We have

$$4(5) = n(S) \cdot n(A) = n(S \times A) = 20.$$

There are twenty intersections in the small town.

Example: In algebra the rectangular or Cartesian coordinate plane is an example of the Cartesian product. We consider the set of all the ordered pairs describing locations in the plane.



Example: A couple is planning their wedding. They have four nieces (Ann, Betty, Cathy, and Deanne) and three nephews (Ed, Fred, and Gill). How many different pairings are possible to have one boy and one girl as a ring bearer and flower girl?

Note that this problem may be considered as either a repeated addition problem or a Cartesian product problem.

Repeated addition: Each niece may be considered to be a set that contains three nephews, so 4(3) = 3 + 3 + 3 + 3 = 12.

Cartesian product: {(A, E), (A, F), (A, G), (B, E), (B, F), (B, G), (C, E), (C, F), (C, G), (D, E), (D, F), (D, G)}

$4(3) = n(\text{nieces}) \cdot n(\text{nephews}) = n(\text{nieces} \times \text{nephews}) = 12$

The couple has twelve choices for one ring bearer and one flower girl.

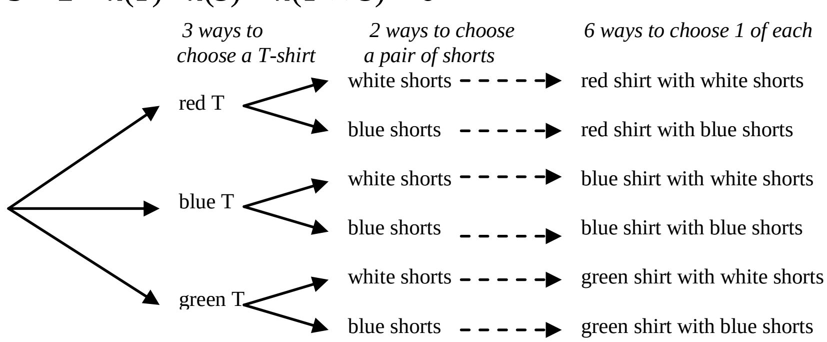
Fundamental Counting Principle

The Cartesian product form for multiplication is the basis for the Fundamental Counting Principle. This principle tells us that if there are "a" ways to do task A and "b" ways to do another task B, there are $a \times b$ ways to do task A followed by task B.

Example: You have three t-shirts and two pairs of shorts. In how many ways can you choose one t-shirt and one pair of shorts to wear?

Answer: We have two distinct sets of objects, t-shirts (T) and shorts (S).

$3 \times 2 = n(T) \cdot n(S) = n(T \times S) = 6$



You have six choices for an outfit of one t-shirt and one pair of shorts.

Notice that the tree diagram for the above problem was easily drawn out as a tree diagram to illustrate all the ordered pairs of t-shirts and shorts. But, for problems with a large number of objects, the Fundamental Counting Principle gives a short-cut way of counting the number of end-branches in a tree diagram without needing to draw out the entire diagram.

Example: A coin is tossed, a die is rolled, and a card is drawn from a standard deck. How many outcomes are possible?

Notice that a tree diagram for this problem would take a lot of time to draw out with all the possibilities. So, we apply the Fundamental Counting Principle to count all the ordered triplets for the coin, die, and card such as (H, 3, ace of hearts) or (T, 2, queen of clubs).

$$n(\text{coin}) \cdot n(\text{die}) \cdot n(\text{card}) = 2 \cdot 6 \cdot 52 = 624$$

There would 624 possible outcomes when a coin is tossed, a die is rolled, and a card is drawn.