Session 12 – Factors, Multiples, and Divisors

How does the following problem relate to the factors in a product?

> *Carla has twelve gum balls and wants to share them among friends where each person receives the same number of gumballs. Carla has several choices for how she shares her gumballs depending on how many friends she shares her gumballs with. What are all the possible ways she can share her gum balls —number of people and number of gumballs each receives?*

Since each person must receive a whole gumball, the problem is asking for all the possible natural number products that can be formed where the product is twelve. That is,

$$1 \times 12,\ 2 \times 6,\ 3 \times 4,\ 4 \times 3,\ 6 \times 2, \text{ and } 12 \times 1.$$

The possibilities are:

She keeps all twelve gumballs, 1(12) = 12.
She and a friend each get six gumballs, 2(6) = 12.
She and two friends each get four gumballs, 3(4) = 12.
She and three friends each get three gumballs, 4(3) = 12.
She and five friends each get two gumballs, 6(2) = 12.
She and eleven friends each get one gumball, 12(1) =2.

We may consider the above problem in three different ways: What are all the ways two natural number *factors* give a product of twelve? What are all the ways we can *multiply* two natural numbers to get twelve? What are the possible natural number *divisors* of twelve that give a natural number quotient? These different perspectives for the above problem motivate the concepts of factors, multiples, and divisors.

Factors, Multiples, and Divisors: Two numbers are *factors* of a number if their product is the number. The number is a *multiple* of a factor. Each factor is a *divisor* of the number.

General Property when the Natural Numbers is the Universal Set:

a is a *factor* of b if there is a k so that $b = ak$ with $\{a, b, k\} \subseteq \boldsymbol{N}$.
b is a *multiple* of a if there is a number k so that $b = ak$ with $\{a, b, k\} \subseteq \boldsymbol{N}$.
a is a *divisor* of b if there is a k so that $b = ak$ with $\{a, b, k\} \subseteq \boldsymbol{N}$.

Numeric Example: Since $5 \times 8 = 40$, both 5 and 8 are *factors* of 40.
Since $5 \times 8 = 40$, 40 is a *multiple* of 5 and 40 is also a *multiple* of 8.
Since $5 \times 8 = 40$, both 5 is a *divisor* of 40 and 8 is also a *divisor* of 40.

Often we need to find all of the factors or multiples of a number. It is convenient to think of this group of factors as a set.

Example: In the introduction problem, the question was asking for all the natural number factors of twelve. The set of factors of twelve, {1, 2, 3, 4, 6, 12}, is a list of possibilities for the number of people who would receive gumballs.

Example: The set of all the whole number factors of 15 is {1, 3, 5, 15}.
The set of all the whole number divisors of 15 is {1, 3, 5, 15}.
The set of all the natural number multiples of 15 is $\{15, 30, 45, 60, \ldots, 15n, \ldots\}$.
The set of all the whole number multiples of 15 is $\{0, 15, 30, 45, 60, \ldots, 15n, \ldots\}$.

Note that the universe affects the answer. Zero is a whole number multiple of every number since $0 \times a = 0$. Also notice that the set of multiples is an infinite set.

Example: $\{x : x$ is a natural number multiple of $4\} = \{4, 8, 12, 16, 20, 24, \ldots, 4n, \ldots\}$
$\{x : x$ is a whole number multiple of $4\} = \{0, 4, 8, 12, 16, 20, 24, \ldots, 4n, \ldots\}$

Example: $\{x : x$ is a natural number factor of $24\} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.
$\{x : x$ is a whole number factor of $24\} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.
$\{x : x$ is a natural number divisor of $24\} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.
$\{x : x$ is a whole number divisor of $24\} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.

Note that the set of factors is the same when the universe is either the natural numbers or the whole numbers.

If we are asked for the set of all factors of a value, we MUST include all the whole number factors for the set to be the correct answer. Notice that the factors generally come in pairs.

$\{1, 2, 3, 4, 6, 8, 12, 24\}$

However, if the product is a perfect square, such as $6 \times 6 = 36$, there is only one factor because it would be paired with itself.

$\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

More on Divisors

Since multiplication and division are inverse operations, the natural number divisors of a value are the same as the factors of that value. It may seem confusing to have two different names for the same set of values, but in some contexts (multiplying contexts) it makes sense to call these values the set of factors, while in other contexts (dividing contexts) it makes sense to call these values divisors.

Even and Odd Numbers: A natural number (or whole number) is an *even number* if it is a multiple of two. A natural number (or whole number) that is not an even number is an *odd number*.

General Property:

A value of the form $2n$, where n is a counting number (or whole number), is an even number.
A value in the form of $2n - 1$ where n is a counting number is an odd number.
A value in the form of $2n + 1$ where n is a whole number is an odd number.

Note that an odd number is always one less (or one more) than some even number, $2n$.

Set-Builder Notation:

The set of even counting numbers is $\{x : x = 2n \text{ where } n \in \boldsymbol{N}\}$.
The set of odd counting numbers is $\{x : x = 2n - 1 \text{ where } n \in \boldsymbol{N}\}$.
The set of even whole numbers is $\{x : x = 2n \text{ where } n \in \boldsymbol{W}\}$.
The set of odd whole numbers is $\{x : x = 2n + 1 \text{ where } n \in \boldsymbol{W}\}$.

Roster Notation:

The set of even counting numbers is {2, 4, 6, 8, 10, …}.
The set of odd counting numbers is {1, 3, 5, 7, 9, …}.
The set of even whole numbers is {0, 2, 4, 6, 8, 10, …}.
The set of odd whole numbers is {1, 3, 5, 7, 9, …}.

Some Other Factor Facts

A counting number that ends in an even digit is an even number.
A counting number that ends in the digit 5 or 0 has 5 as a factor.
A counting number that ends in the digit 0 has 10 as a factor.
A counting number that ends in two zeros has 100 is a factor.

Examples with Sets

When working with sets, mathematical "or" corresponds to the set operation *union.*

Example: { *x* : *x* is a factor of 12 **or** *x* is a factor of 10}
= { *x* : *x* is a factor of 12} $\cup$ { *x* : *x* is a factor of 10}
= {1, 2, 3, 4, 6, 12} $\cup$ {1, 2, 5, 10}
= {1, 2, 3 ,4, 5, 6, 10, 12}

When working with sets, mathematical "and" corresponds to the set operation *intersection.*

Example: { *x* : *x* is a factor of 12 **and** *x* is a factor of 10}
= { *x* : *x* is an factor of 12} $\cap$ { *x* : *x* is a factor of 10}
= {1, 2, 3, 4, 6, 12} $\cap$ {1, 2, 5, 10}
= {1, 2}