Session 13 – Exponents and Simplifying Expressions

Which would you choose?

You are given an option of being paid either

- *$1,000,000 or*
- *for thirty days receiving one cent the first day and each succeeding day double the amount of the previous day.*

How much would you receive on the 30th day?

The double of an amount is the same as multiplying an amount by two. So, the daily amounts in cents for each day would be:

Day 1 1¢
Day 2 2¢
Day 3 $2 \times 2 = 4$¢
Day 4 $2 \times 2 \times 2 = 8$¢
Day 5 $2 \times 2 \times 2 \times 2 = 16$¢
Day 6 $2 \times 2 \times 2 \times 2 \times 2 = 32$¢
Day 7 $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$¢

This is getting to be a lot of multiplications to write. We use exponents to write the problem for each day in a shorter form. Since Day 7 has two used as a factor six times, we may write the problem as

Day 7 $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$¢

Then Day 8 would use another factor of 2 for

Day 8 $2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$¢.
Day 9 $2^8 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$¢
Day 10 $2^9 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512$¢

We note that the number of two's used as a factor is one less than the number of the day, so

Day 30 2^{29}.

This says that we would need to multiply 29 two's to find the amount of money in cents for the thirtieth day. Determine this amount then decide which choice would be the best choice.

The above problem motivates why exponents are used as a shortcut to setup problems. We now give more examples involving exponents.

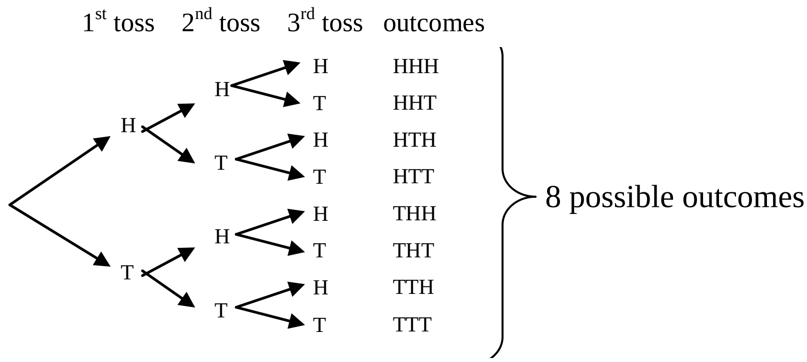
Exponents

***Definition of an Exponent*:** The expression $\underbrace{b \times b \times \cdots \times b}_{n}$, which is n factors of b used for $n-1$ multiplications, can be written as b^n. The b is called the base and the n is called the exponent.

Examples:

1. A coin is flipped three times. How many outcomes are possible?

   To solve the above problem in Session 9, we completed a tree diagram (as illustrated below) to solve the problem to motivate the Fundamental Counting Principle



$2 \times 2 \times 2 = 8$ WAYS

Note that we could also express the problem in exponential form as

$$2^3 = 2 \times 2 \times 2 = 8.$$

We have eight possible outcomes when a coin is flipped three times.

2. What if we were to flip a coin four times? How many outcomes are possible?

   Setting up the problem using exponents, we obtain $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
   There would be sixteen possible outcomes if a coin is flipped four times.

3. If we flip the coin thirteen times, then

$$2^{13} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 8{,}192.$$

   We would have 8,192 different possible outcomes if a coin is flipped thirteen times.

Note that as the number of times we have to repeat the multiplication becomes larger, the notation of using only multiplication becomes cumbersome. So, we use the short-cut notation with exponents to represent this type of repeated multiplication.

Example: In $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, the 2 is repeated 13 times.

Using exponent notation, we can write this as 2^{13}

The small, raised 13 is the exponent.

The 2 is called the "base."

More Examples:

1. $3^2 = 3 \times 3 = 9$

2. Use repeated multiplication to find 5^3, 10^5, and 9^3.

$5^3 = 5 \times 5 \times 5 = 25 \times 5 = 125$
$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$
$9^3 = 9 \times 9 \times 9 = 9 \times 81 = 729$

3. Dino is eating at his favorite Italian restaurant. On the menu there are three wines, three salads, three entrees, and three desserts. There are also three credit cards in his wallet. How many ways can he choose a meal choosing exactly one of each type of item and pay for it with a credit card?

We apply the Fundamental Counting Principle and use exponents to obtain
$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$
Dino has 243 choices for a meal and paying for it.

Exponents and Cartesian Product

Since exponents represent repeated multiplication, exponents may also be used to represent Cartesian products of sets (the set of ordered pairs). For example, we may write $A \times A$ as A^2.

Examples:

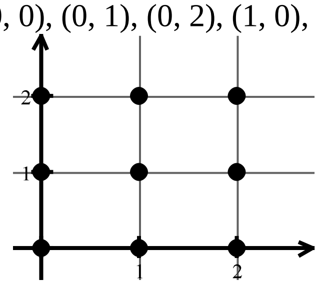
1. A coin is flipped twice. What are the possible outcomes?

   We may solve this problem as a Cartesian product question by giving the set all the possible ordered pairs. Let C = {H, T}. Then
   $$C^2 = C \times C = \{(H, H), (H, T), (T, H), (T,T)\}.$$

2. If set $A = \{1, 2\}$, then $A^2 = A \times A = \{(1, 1), (1, 2), (2, 1)\}$.

3. Let $B = \{0, 1, 2\}$. Determine the number of elements in B^2, check the solution by finding B^2, and then plot B^2 on a coordinate plane.

   The number of elements is $(n(B))^2 = 3^2 = 9$.
   Reminder: $n(B)$ is the cardinal number for B.
   We have $B2 = B \times B = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.



4. Let D be the set of possible outcomes when rolling one 6-sided die. Write D using roster notation. Write out D^2 using roster notation. What does D^2 represent? How many elements are in D^2?

   Solutions: $D = \{1, 2, 3, 4, 5, 6\}$.
   $D^2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$
   $(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$
   $(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

   D^2 represents the possible outcomes of rolling a die twice or of rolling two dice.
   There are $6^2 = 36$ elements in D^2.

Scientific Notation

Motivation Problem

The mass of a mass of the earth is approximately 6,000,000,000,000,000,000,000,000 kilograms. Scientists would write this value in scientific notation using powers of ten or exponential form. A scientist would write the mass of the earth as 6×10^{24} *kilograms.*

Scientific notation is useful for writing very large or very small numbers. For now, we will focus on using scientific notation to write very large numbers. Consider the value three million. We typically write this number as 3,000,000, but we may also write it in *scientific notation*:

$$3{,}000{,}000 = 3 \times 1{,}000{,}000 = 3 \times 10^6.$$

Notice that the exponent is the same as the number of zeros following the 3. Three million is a verbal expression, 3,000,000 is a decimal expression, and 3×10^6 is scientific notation.

When we reviewed place value, each column of the place value chart had a place value 10 times as much as the column to the right of it. We may use *scientific notation* to simplify writing the values in the place value chart.

×10 ×10 ×10 ×10 ×10 ×10 ×10 ×10

10^8	10^7	10^6	10^5	10^4	10^3	10^2	10^1	1
hundred million$	ten million$	million$						

We now know three ways to express this type of number: as a standard numeral, in verbal form, and in scientific notation. Each form has its own advantages.

Examples:

Standard Numeral	**Verbal Form**	**Scientific Notation**
5,000,000,000	five billion	5×10^9
60,000,000	sixty million	6×10^7
70,000	seventy thousand	7×10^4
120,000,000	one hundred twenty million	12×10^7
17,000,000,000	seventeen billion	17×10^9

Since we are only working with whole numbers, we are not writing the scientific notation with a decimal point. Later after we have worked with decimals, we will revisit scientific notation and use the normalized form for scientific notation.

Multiplying Variable Expressions

Previously we learned to add variable expressions by *adding like terms*. Now that we have exponents, we can also multiply variable expressions. First we note that $x \cdot x = x^2$ since the variable x

has been used as a factor twice. This is just applying the definition of an exponent. Similarly, we have that $a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6$.

Now consider $3x \cdot 2x$. Since everything in this expression is multiplied, we use the commutative and associative properties of multiplication with the definition of exponent to find the product:

$(3x) \cdot (2x) = 3 \cdot (x \cdot 2) \cdot x$	Associative Property of Multiplication
$= 3 \cdot (2 \cdot x) \cdot x$	Commutative Property of Multiplication
$= (3 \cdot 2) \cdot (x \cdot x)$	Associative Property of Multiplication
$= 6 \cdot x^2$	Arithmetic and Definition of Exponent
$= 6x^2$	Simplified way of writing multiplication

Example: Multiply $(4x)(2x)(x)$.

$$\begin{aligned} 4x \cdot 2x \cdot x &= (4 \cdot 2 \cdot 1) \cdot (x \cdot x \cdot x) \\ &= \quad 8 \quad \cdot \quad x^3 \\ &= 8x^3 \end{aligned}$$

Example: Multiply $(3x)(y)(x2)(5y)$.

$$\begin{aligned} 3x \cdot y \cdot x^2 \cdot 5y &= (3 \cdot 5) \cdot (x \cdot x^2) \cdot (y \cdot y) \\ &= (15) \cdot (x \cdot x \cdot x) \cdot (y \cdot y) \\ &= 15 \quad \cdot \quad x^3 \quad \cdot \quad y^2 \\ &= 15x^3y^2 \end{aligned}$$

Notice that coefficients are grouped together and then each different variable is grouped together.

We note that we *only add like terms and obtain like terms*. But when we multiply expression with variables we do not obtain ***like terms***, instead we get expressions with exponents. It is important to learn to make this distinction when simplifying variable expressions.

Compare each of the following when adding or multiplying the same terms.

Addition	**Multiplication**
$2x + 3x = 5x$	$2x \cdot 3x = 6x^2$
$4y + y + 2y = (4+1+2)y$ $= 7y$	$4y \cdot y \cdot 2y = (4 \cdot 1 \cdot 2)(y \cdot y \cdot y)$ $= 8y^3$
$a + 2b + a + b = a + a + 2b + b$ $= 2a + 3b$	$a \cdot 2b \cdot a \cdot b = 2 \cdot a \cdot a \cdot b \cdot b$ $= 2a^2b^2$

Order of Operations

Motivation Problem

Find the value of 5 + 6(4).

Unless we have specific rules for finding the above value, different people could obtain different answers. Some may multiply first then add to obtain 29 and another person might add first then multiple to obtain 44. ***This situation is unacceptable***. So, people have agreed on certain standard rules for determining the value of expressions that involve different operations. The most common rules and the ones that we are going to use are called the *algebraic order of operations*.

We have learned the operations involving exponents, division, multiplication, subtraction and addition. In order to perform complex computations with these operations properly, we need to perform these operations in a particular order. The standard rules for the algebraic order of operations are:

First, we perform operations that are grouped such as by Parentheses.
Second, we compute Exponents.
Third, we perform the Multiplication and Division from left to right.
Finally, we perform the Addition and Subtraction from left to right.

The acronym for remembering the order of operations is PEMDAS, which stands for Parentheses, Exponents, Multiplication, Division, Addition and Subtraction. A mnemonic device for remembering this acronym is ***P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally*.

Example: The opening problem 5 + 6(4) would be worked as follows.

$$\begin{aligned} 5 + 6(4) &= 5 + 24 \qquad \text{since multiplication comes before addition} \\ &= 29 \end{aligned}$$

A common mistake associated with this acronym is to forget that PEMDAS does not properly reflect the "left-to-right" part of the rules in the first paragraph. Multiplication and division have the same priority, and are done left-to-right. Addition and subtraction have the same priority, and are done left-to-right. A more accurate mnemonic might be PE <u>MD</u> <u>AS</u> to remind yourself that the underlined pairs are done together, moving left-to-right.

That is, <u>multiplication and division</u> are done <u>in the same step</u>, and they are <u>done left-to-right</u>.

Example: In 100 ÷ 4 × 5, the multiplication and division must be done left-to-right, which means that in this case the division is actually done ***before*** the multiplication.

$$\begin{aligned} 100 \div 4 \times 5 &= 25 \times 5 \\ &= 125 \end{aligned}$$

In the same way, <u>addition and subtraction</u> are done <u>in the same step</u>, and they are also <u>done left-to-right</u>.

Example: In 20 – 2 + 8, the addition and subtraction must be done left-to-right, which means in this case the subtraction is actually done ***before*** the addition:

$$20 - 2 + 8 = 18 + 8$$
$$= 26$$

Examples: In each of the following the step to be done next is underlined.

$12 + 15 \div 5 - 2$
$= 12 + \underline{15 \div 5} - 2$
$= \underline{12 + 3} - 2$
$= \underline{15 - 2}$
$= 13$

$2^3 \times 10 - 3 + 9 \div 3$
$= \underline{2^3} \times 10 - 3 + 9 \div 3$
$= \underline{8 \times 10} - 3 + \underline{9 \div 3}$
$= \underline{80 - 3} + 3$
$= \underline{77 + 3}$
$= 80$

$3(10^2 - 4 \times 5^2) + 15$
$= 3(\underline{10^2} - 4 \times \underline{5^2}) + 15$
$= 3(100 - \underline{4 \times 25}) + 15$
$= 3\underline{(100 - 100)} + 15$
$= \underline{3(0)} + 15$
$= \underline{0 + 15}$
$= 15$

More Examples Multiplying Expression

Consider $(2x)^4$. In this case we are taking the entire product inside the parentheses to the 4th power.

Steps	*Reasons*
$(2x)^4 = (2x)(2x)(2x)(2x)$	Definition of the exponent 4
$= (2 \cdot 2 \cdot 2 \cdot 2)(x \cdot x \cdot x \cdot x)$	Commutative and Associative Properties
$= 2^4x^4$	Definition of Exponents

Here is another example: $(ab^2)^3$.

Steps	*Reasons*
$(ab^2)^3 = (ab^2) \cdot (ab^2) \cdot (ab^2)$	Definition of the exponent 3
$= (a \cdot a \cdot a) \cdot (b^2 \cdot b^2 \cdot b^2)$	Commutative and Associative Properties
$= (a \cdot a \cdot a) \cdot (b \cdot b) \cdot (b \cdot b) \cdot (b \cdot b)$	Definition of the exponent 2
$= (a \cdot a \cdot a) \cdot (b \cdot b \cdot b \cdot b \cdot b \cdot b)$	Associative Property
$= (a^3)(b^6)$	Definition of Exponents

Another example: $(x^4y^3)^2$.

Steps	*Reasons*
$(x^4y^3)^2 = (x^4y^3) \cdot (x^4y^3)$	Definition of the exponent 2
$= (x \cdot x \cdot x \cdot x)\ (y \cdot y \cdot y)\ (x \cdot x \cdot x \cdot x)\ (y \cdot y \cdot y)$	Definition of the exponents 4 and 3
$= (x \cdot x \cdot x \cdot x)\ (x \cdot x \cdot x \cdot x)\ (y \cdot y \cdot y)\ (y \cdot y \cdot y)$	Commutative Property of Multiplication
$= (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)\ (y \cdot y \cdot y \cdot y \cdot y \cdot y)$	Associative Property of Multiplication
$= x^8y^6$	Definition of Exponents

Note that these problems are motivating the Properties of Exponents, which are methods for working these problems in a more efficient manner. Some of the Properties of Exponents will be developed later in Session 29.