Session 21 –Fraction Addition and Subtraction and Mixed Number Notation

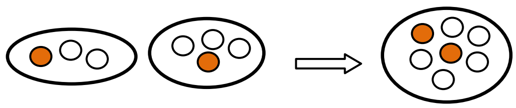
Solve and compare the following two problems.

Kim made one out of three free throws in a game and one out of four free throws in the next game. What fractional part of the free throws did Kim make?

Pat walked one-third of a mile in the morning and one-fourth of a mile in the evening. What fractional part of a mile did Pat walk?

We begin with these two examples to illustrate one of the reasons many people have problems with addition and subtraction of fractions.

Example from above: Kim made one out of three free throws in a game and one out of four free throws in the next game. What fractional part of the free throws did Kim make?

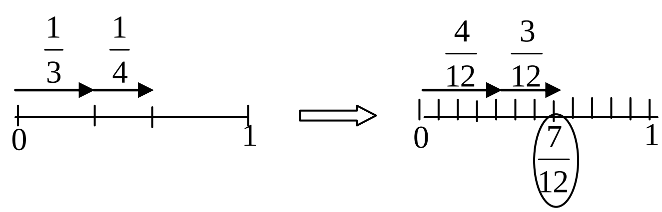


$$\frac{1}{3} \oplus \frac{1}{4} = \frac{2}{7}$$

Kim made two-sevenths of the free throws, or Kim made two out of seven free throws.

Important Note. Here, we did a union of two disjoint sets as with addition of whole numbers. But we do ***not*** consider this to be the addition of two fractions since the size of the whole is different for each fraction. Each fraction is referencing a different sized whole object. We used the symbol $\oplus$ to indicate that it is a different type of addition and ***not*** the addition of fractions. This also shows a common error in adding fractions that many people make, which is adding both the numerators and the denominators.

Example from above: Pat walked one-third of a mile in the morning and one-fourth of a mile in the evening. What fractional part of a mile did Pat walk?



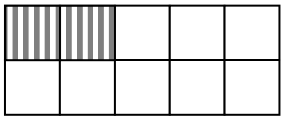
$$\frac{1}{3} + \frac{1}{4} = \frac{1 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

Pat walked seven-twelfths of a mile.

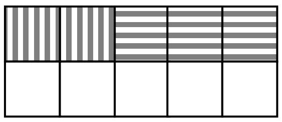
Note. Each of the above problems is a type of addition. The first problem is ***not*** the addition of fractions, because the size of the whole is different for each fraction. But the second problem is an example of how we will define addition of fractions where the size of the whole is the same for each fraction.

Addition and Subtraction of Fractions with Common Denominators

We begin with adding $\frac{2}{10}+\frac{3}{10}$ by using a rectangle to represent the whole and divided into ten equal-sized pieces.



First, we shade $\frac{2}{10}$ of a whole rectangle.



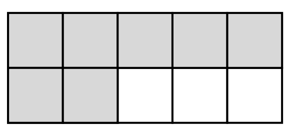
Next, to show the addition, we shade $\frac{3}{10}$ more.

We now have $\frac{5}{10}$ of the whole rectangle shaded. So, we conclude that $\frac{2}{10}+\frac{3}{10}=\frac{5}{10}$.

Notice that we added tenths to tenths and our answer was in tenths. It appears that the rule for adding fractions that have the same denominator is to add the numerators and keep the common denominator.

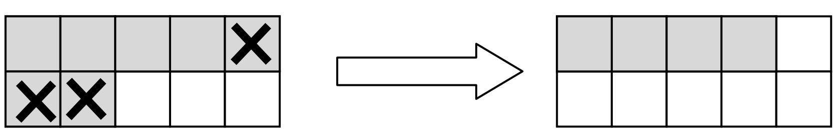
We can still simplify the answer $\frac{5}{10}$ as in the previous sessions by using the Fundamental Law of Fractions $\frac{5}{10}=\frac{1\cdot 5}{2\cdot 5}=\frac{1}{2}$ or by division by the greatest common factor $\frac{5}{10}=\frac{5\div 5}{10\div 5}=\frac{1}{2}$. So, we have $\frac{2}{10}+\frac{3}{10}=\frac{5}{10}=\frac{1}{2}$.

Similarly, we subtract $\frac{7}{10}-\frac{3}{10}$ by using a rectangle to represent the whole with 7 parts shaded.



This represents the $\frac{7}{10}$ of a whole from which we will take $\frac{3}{10}$ of the whole rectangle away.

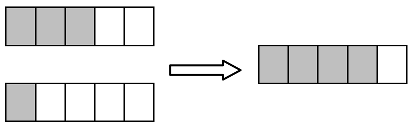
Next, we subtract $\frac{3}{10}$ of the whole rectangle by crossing out or unshading 3 of the parts.



We conclude that $\frac{7}{10}-\frac{3}{10}=\frac{4}{10}$ which can be simplified to $\frac{2}{5}$.

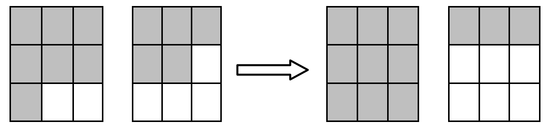
Here are two more examples with models.

Use fraction strips for $\frac{3}{5}+\frac{1}{5}$



$$\frac{3}{5}+\frac{1}{5} = \frac{4}{5}$$

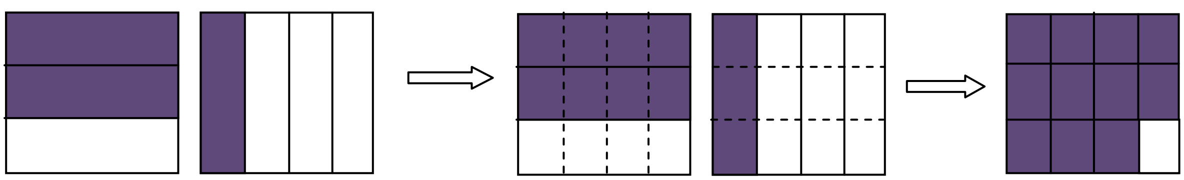
Use an area model for $\frac{7}{9}+\frac{5}{9}$



$$\frac{7}{9}+\frac{5}{9} = \frac{12}{9}=1\frac{3}{9}=1\frac{1}{3}$$

Addition and Subtraction of Fractions with Unequal Denominators

Now how can we add $\frac{2}{3}+\frac{1}{4}$? As with problems with common denominators, we can draw a diagram. We have a problem since all the pieces are not the same size. Though, after cutting each piece to make pieces of the same size, we are able to work the problem. In other words, we change the problem so that the fractions are equivalent fractions with common denominators as illustrated below.



$$\frac{2}{3}+\frac{1}{4} = \frac{2\cdot 4}{3\cdot 4}+\frac{1\cdot 3}{4\cdot 3}=\frac{8}{12}+\frac{3}{12} = \frac{8+3}{12}=\frac{11}{12}$$

Note that we cut each third into fourths and each fourth into thirds so that each small piece represents one-twelfth of a whole square. Further, we note that the LCM (3, 4) = 12, so we have chosen 12 for the least common denominator.

Addition and Subtraction Rule

In summary, to add or subtract fractions, it is sufficient to change the fractions so that they have common denominators. Then we add or subtract the numerators and keep the common denominator. Finally, we simplify that answer, if it is not already in simplest form.

Here is an example adding more than two fractions at a time. Find the sum $\frac{2}{5}+\frac{1}{10}+\frac{1}{2}+\frac{4}{25}$.

First, find the least common multiple of denominators. This will give us the least common denominator. We find LCM (5, 10, 2, 25) by the prime factorization method.

$5 = 5$, $10 = 2 \cdot 5$, $2 = 2$, and $25 = 5^2$
So the LCM (5, 10, 2, 25) = $2 \cdot 5^2 = 2 \cdot 25 = 50$

We now change each addend to fractions with the common denominator.

$$\frac{2}{5} + \frac{1}{10} + \frac{1}{2} + \frac{4}{25} = \frac{2\cdot 10}{5\cdot 10} + \frac{1\cdot 5}{10\cdot 5} + \frac{1\cdot 25}{2\cdot 25} + \frac{4\cdot 2}{25\cdot 2}$$

$$= \frac{20}{50} + \frac{5}{50} + \frac{25}{50} + \frac{8}{50}$$

$$= \frac{20+5+25+8}{50}$$

$$= \frac{58}{50} = \frac{29}{25} = 1\frac{4}{25}$$

Note that we simplified $\frac{58}{50}$ and then changed it to a mixed number. A more detailed illustration follows where we show each step: first simplify the fraction, then since the fraction is an improper fraction, split the improper fraction into a whole and a fractional part, and then write as a mixed number.

$$\frac{58}{50} = \frac{29\cdot 2}{25\cdot 2} = \frac{29}{25}$$

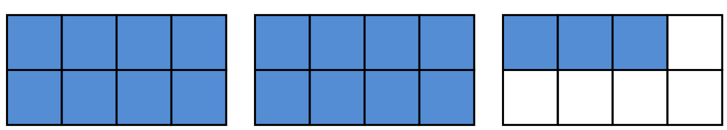
$$= \frac{25}{25} + \frac{4}{25} = 1 + \frac{4}{25}$$

$$= 1\frac{4}{25}$$

This answer, $1\frac{4}{25}$, is a *mixed number*. Any improper fraction can be also be expressed as a mixed number because there is more than one whole in the improper fraction.

Changing an Improper Fraction to a Mixed Number

Consider the following illustration where each rectangle represents one whole and each rectangle is cut into eight pieces of the same size.



The illustration shows that the improper fraction $\frac{19}{8}$. Also, the illustration has two whole rectangles and three-eighths of another rectangle, that is, it shows the mixed number $2\frac{3}{8}$. So, we have illustrated that $\frac{19}{8} = 2\frac{3}{8}$. The model also motivates a method for changing from an improper fraction to a mixed number. Since each group of 8 pieces is a whole piece, we could change to a mixed number by dividing 19 by 8 to obtain two whole pieces and three pieces remaining.

$$19 \div 8 = 2\frac{3}{8}$$

The problem shows that we can think of a fraction as another way to represent division, $\frac{a}{b} = a \div b$ or as $b\overline{)a}$. For example, we may change an improper fraction like $\frac{13}{5}$ to a mixed number by division where we interpret the fraction as division. We will write the remainder as a fraction.

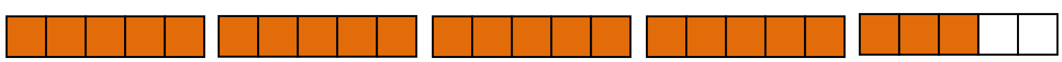
$$\frac{13}{5} \quad = \quad 13 \div 5 \qquad \begin{array}{r} 2 \\ 5\overline{)13} \\ -10 \\ \hline 3 \end{array}$$

We take the remainder of 3 and write it as $\frac{3}{5}$ of another whole giving us $2\frac{3}{5}$. So $\frac{13}{5} = 2\frac{3}{5}$.

Changing a Mixed Number to an Improper Fraction

Suppose we need to change $4\frac{3}{5}$ into an improper fraction. (We will need to do this when we start multiplying and dividing fractions.) We illustrate where each rectangle is one whole.

Now divide each whole rectangle into five equal parts to illustrate $\frac{23}{5}$.

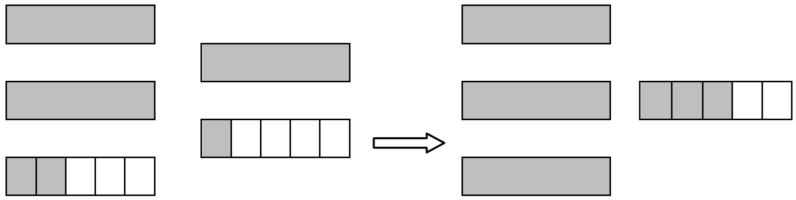


We have shown that $4\frac{3}{5} = 1+1+1+1+\frac{3}{5} = \frac{5}{5}+\frac{5}{5}+\frac{5}{5}+\frac{5}{5}+\frac{3}{5} = \frac{4\cdot 5}{5}+\frac{3}{5} = \frac{23}{5}$.
Notice that the process for finding the number of fifths is to cut each of the four wholes into five-fifths and add the three-fifths for a total of twenty-three fifths, i.e., $4\frac{3}{5} = \frac{4\cdot 5+3}{5} = \frac{23}{5}$.

Addition and Subtraction of Mixed Numbers

The addition and subtraction of mixed numbers may be accomplished in the same manner as we added and subtracted whole numbers. In other words, we add the values by corresponding place-value with mixed numbers that means we add or subtract the whole number portions and the fraction portions separately making exchanges when needed. We demonstrate the process with models in the following examples.

Example: Find $2\frac{2}{5}+1\frac{1}{5}$.



$$2\frac{2}{5} \quad + \quad 1\frac{1}{5} \quad = \quad 3\frac{3}{5}$$

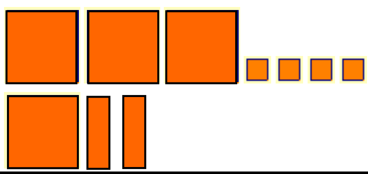
Note that we added the whole and the fraction portions separately.

Example: Find $3\frac{4}{9}+1\frac{2}{3}$.

We first change the fraction portions to a common denominator.

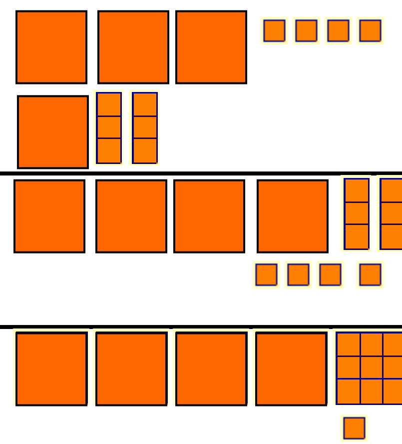
$3\frac{4}{9}$

$+1\frac{2}{3}$



$= 3\frac{4}{9}$

$= +1\frac{6}{9}$



We add the whole and the fraction portions separately.

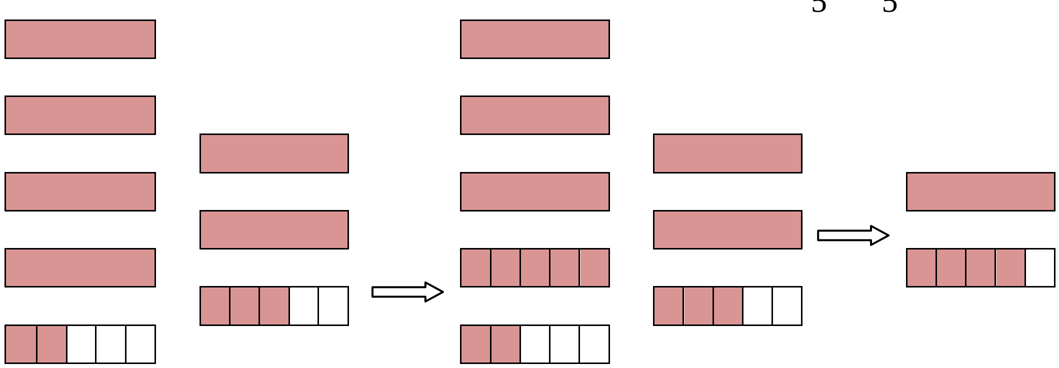
$4\frac{10}{9}$

We change the improper fraction to a mixed number $4\frac{10}{9} = 4+\frac{10}{9} = 4+1\frac{1}{9} = 5\frac{1}{9}$.

$= 5\frac{1}{9}$

Example: Find $4\frac{2}{5}-2\frac{3}{5}$.

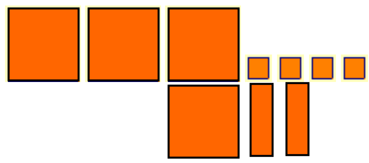
Here we need to make an exchange in order to subtract the fractional parts. So, we need to cut one of the four whole rectangles into fifths, e.g., $4\frac{2}{5}=3\frac{7}{5}$.



$$4\frac{2}{5} \quad - \quad 2\frac{3}{5} \quad = \quad 3\frac{7}{5} \quad - \quad 2\frac{3}{5} \quad = \quad 1\frac{4}{5}$$

Example: Find $3\frac{4}{9}-1\frac{2}{3}$

$$3\frac{4}{9}$$
$$-1\frac{2}{3}$$



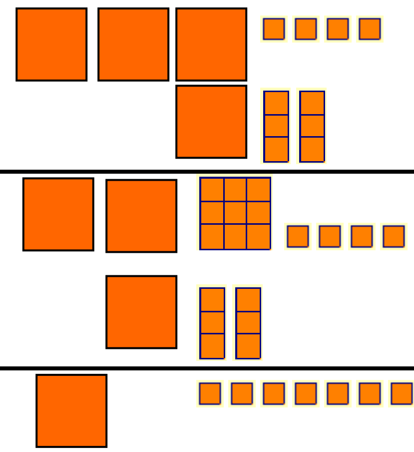
We need to make an exchange. We cut one whole square into nine equal parts.

That is, $3\frac{4}{9}=2+1\frac{4}{9}=2\frac{13}{9}$.

We subtract the whole and fraction part separately.

We first change the fraction portions to a common denominator.

$$= 3\frac{4}{9}$$
$$= -1\frac{6}{9}$$
$$2\frac{13}{9}$$
$$-1\frac{6}{9}$$
$$= 1\frac{7}{9}$$



Mixed Numbers Versus Improper Fractions

Note. It is often much easier to add and subtract mixed numbers without converting to improper fractions. Changing to improper fractions increases the number of computations and makes many problems harder to simplify.

Example. Compute $12\frac{13}{15}+7\frac{8}{21}$ two ways: as mixed numbers and as improper fractions

As mixed numbers

$$12\frac{13}{15}+7\frac{8}{21} = 12\frac{91}{105}+7\frac{40}{105}$$ Find the common denominator.

$$= 19\frac{131}{105} = 20\frac{26}{105}$$ Change to a mixed number.

As improper fractions

$$12\frac{13}{15}+7\frac{8}{21} = \frac{193}{15}+\frac{155}{21}$$ Change to an improper fraction.

$$= \frac{1351}{105}+\frac{775}{105}$$ Find the common denominator.

$$= \frac{2126}{105} = 20\frac{26}{105}$$ Change to a mixed number.

The multiplying and dividing of the large values greatly increases the chance of error. Also, if the fractions needed to be simplified, the simplification with the large values would be much more difficult when using the improper fractions method.