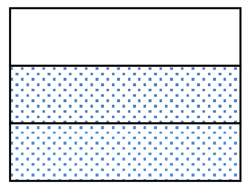
Session 22 –Fraction Multiplication

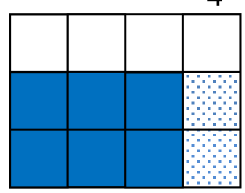
Solve the following problem.

Pat used three-fourths of a bag of flour that was two-thirds full. How much of a full bag of flour did Pat use.

Here, we solve this problem by the use of an area model where we will use a rectangle to represent a bag of flour. We shade $\frac{2}{3}$ of the rectangle to show that we have two-thirds of a bag of flour.

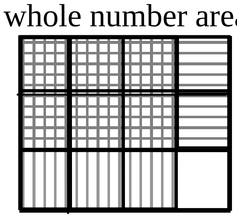


Next, since Pat used $\frac{3}{4}$ of the $\frac{2}{3}$ bag of flour, we shade $\frac{3}{4}$ of the $\frac{2}{3}$ of the rectangle.



We now have $\frac{6}{12} = \frac{1}{2}$ of the rectangle that has been shaded twice; this is the region that represents the amount of flour used by Pat. Therefore, Pat used one-half of a full bag of flour.

Note that this problem is an example of $\frac{3}{4} \cdot \frac{2}{3}$. We could have also illustrated the problem with the following model which builds on the whole number area model for multiplication.



The diagram has $\frac{2}{3}$ shaded horizontally and $\frac{3}{4}$ shaded vertically. The solution is the amount of area shaded both horizontally and vertically.

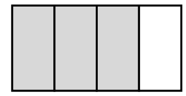
Note that we have completed the multiplication problem: $\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{1}{2}$.

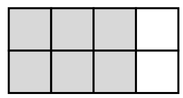
Multiplication of Fractions

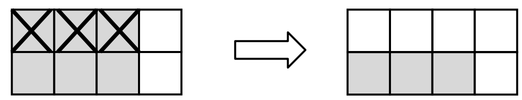
Example: *A recipe calls for* $\frac{3}{4}$ *cup of flour. If you are making only* $\frac{1}{2}$ *the recipe, how much flour do you use?*

We can draw a diagram to work this problem, but it will take us several steps to work it out.

We begin with a rectangle as our whole to represent a full cup of flour. 

Then to represent $\frac{3}{4}$ of a cup of flour we divide the rectangle into 4 equal pieces and shade 3 of them. The shaded part is $\frac{3}{4}$ of the whole. 

So how do we take half of this shaded part of the rectangle when there are 3 shaded parts?
One way would be to split the rectangle in half horizontally.
Notice that we still have the same part of the rectangle shaded, but it is now divided into eighths. 

Finally, since we want one-half of the partial cup of flour, we choose half of the shaded boxes. This leaves $\frac{3}{8}$ of the rectangle shaded. 

We conclude that $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$.

A Word of Caution: It is important to notice that in this problem we have taken *half of* and have *not divided by* $\frac{1}{2}$. Remember that *of* is usually translated as multiplication. What we have computed is $\frac{1}{2}\cdot\frac{3}{4}=\frac{3}{8}$.

Rule for Multiplying Fractions: A more efficient way of multiplying two fractions is to realize that $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$, that is, we can multiply the numerators to get the numerator of the product and we can multiply the denominators to get the denominator of the product. Even better, if we remove all the common factors from the numerator and denominator before we multiply; our final answer will be in simplest form. That is, we can simplify by dividing common factors where for any common factor one is divided from the numerator and one is divided from the denominator.

Example: Evaluate $\frac{5}{9}\cdot\frac{7}{13}$.

$$\frac{5}{9}\cdot\frac{7}{13}=\frac{5\cdot 7}{9\cdot 13}=\frac{35}{117}$$

Example: Evaluate $\frac{5}{6}\cdot\frac{14}{15}$.

$$\frac{5}{6}\cdot\frac{14}{15}=\frac{5\cdot 14}{6\cdot 15}=\frac{70}{90}=\frac{7\cdot 10}{9\cdot 10}=\frac{7}{9} \quad \text{or} \quad \frac{5}{6}\cdot\frac{14}{15}=\frac{\overset{1}{\cancel{5}}}{3\cdot\underset{1}{\cancel{2}}}\cdot\frac{7\cdot\overset{1}{\cancel{2}}}{\underset{1}{\cancel{5}}\cdot 3}=\frac{7}{9}$$

Two Methods of Simplifying

1. Prime Factorization.

$$\frac{15}{8}\cdot\frac{4}{9} = \frac{3\cdot 5}{2\cdot 2\cdot 2}\cdot\frac{2\cdot 2}{3\cdot 3}$$

$$= \frac{\overset{1}{\cancel{3}}\cdot 5}{\underset{1}{\cancel{2}}\cdot \underset{1}{\cancel{2}}\cdot 2}\cdot\frac{\overset{1}{\cancel{2}}\cdot \overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}\cdot 3}$$

$$= \frac{5}{2\cdot 3} = \frac{5}{6}$$

or $\Longleftrightarrow$

2. Divide by Common Factors

Divide the 4 and 8 both by 4.

$$\frac{15}{8}\cdot\frac{4}{9} = \frac{\overset{5}{\cancel{15}}}{\underset{2}{\cancel{8}}}\cdot\frac{\overset{1}{\cancel{4}}}{\underset{3}{\cancel{9}}} = \frac{5}{6}$$

Divide the 15 and 9 both by 3.

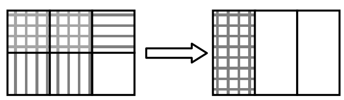
Two Final Reminders about Simplifying by Dividing Common Factors

1. We may only simplify by dividing common factors when multiplying fractions. We cannot simplify by dividing common factors when adding or subtracting fractions. We use basic examples with models to help understand this rule.

   Example:

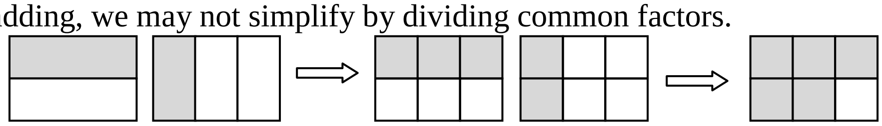
   Note here when we are multiplying, we may simplify by dividing common factors.

$$\frac{1}{2}\cdot\frac{2}{3} = \frac{1\cdot\overset{1}{\cancel{2}}}{3\cdot\underset{1}{\cancel{2}}} = \frac{1}{\underset{1}{\cancel{2}}}\cdot\frac{\overset{1}{\cancel{2}}}{3} = \frac{1}{3}$$



   Note here when we are adding, we may not simplify by dividing common factors.

$$\frac{1}{2}+\frac{2}{3} = \frac{3}{6}+\frac{4}{6} = \frac{7}{6} = 1\frac{1}{6}$$



   We must always use one factor in the numerator and one factor in the denominator when dividing common factors because it is the division bar (fraction bar) that we are interpreting as *divided by*.

Multiplication with Mixed Numbers

To multiply a fraction or a mixed number by a mixed number or whole number, we usually rewrite each term that is a mixed number or whole number as an equivalent improper fraction.

Examples:

$$4\frac{1}{2}\cdot\frac{2}{15} = \frac{9}{2}\cdot\frac{2}{15}$$

$$= \frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{2}}}\cdot\frac{\overset{1}{\cancel{2}}}{\underset{5}{\cancel{15}}}$$

$$= \frac{3}{5}$$

$$3\frac{3}{8}\cdot 4 = \frac{27}{8}\cdot\frac{4}{1}$$

$$= \frac{27}{\underset{2}{\cancel{8}}}\cdot\frac{\overset{1}{\cancel{4}}}{1}$$

$$= \frac{27}{2} = 13\frac{1}{2}$$

Mixed Numbers Versus Improper Fractions

When we added mixed numbers, we found that it is usually easier to leave each addend as a mixed number and *not* change to improper fractions (see the previous session 21); whereas, when multiplying fractions it is usually easier to change to improper fractions and *not* leave as mixed numbers. We illustrate with the examples below.

Example: Compute $2\frac{1}{3} \cdot 1\frac{1}{4}$.

We use the distributive property of multiplication over addition to compute the multiplication without changing the mixed numbers to improper fractions.

$$\begin{aligned}
2\frac{1}{3} \cdot 1\frac{1}{4} &= \left(2\frac{1}{3}\right) \cdot \left(1 + \frac{1}{4}\right) \\
&= \left(2\frac{1}{3}\right) \cdot 1 + \left(2\frac{1}{3}\right) \cdot \frac{1}{4} \\
&= \left(2 + \frac{1}{3}\right) \cdot 1 + \left(2 + \frac{1}{3}\right) \cdot \frac{1}{4} \\
&= 2 \cdot 1 + \frac{1}{3} \cdot 1 + 2 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} \\
&= 2 + \frac{1}{3} + \frac{1}{2} + \frac{1}{12} \\
&= 2 + \frac{4}{12} + \frac{6}{12} + \frac{1}{12} \\
&= 2\frac{11}{12}
\end{aligned}$$

Now we rework the problem by first changing each mixed number into an improper fraction.

$$2\frac{1}{3} \cdot 1\frac{1}{4} = \frac{7}{3} \cdot \frac{5}{4} = \frac{35}{12} = 2\frac{11}{12}$$

Note the fewer number of operations needed. Also, note that with the mixed number form we needed to find a common denominator to add the fractions. For some problems, the mixed number form is easier to use when making mental computations, but usually it is easier to work multiplication problems involving mixed numbers by changing to improper fractions. The opposite was true when adding and subtracting mixed numbers.

Example: Evaluate $3\frac{1}{2} \cdot 2\frac{2}{3}$.

Again, we use the distributive property of multiplication over addition to compute the multiplication without changing the mixed numbers to improper fractions.

$$
\begin{aligned}
3\frac{1}{2}\cdot 2\frac{2}{3} &= \left(3\frac{1}{2}\right)\cdot\left(2+\frac{2}{3}\right)\\
&= \left(3\frac{1}{2}\right)\cdot 2+\left(3\frac{1}{2}\right)\cdot\frac{2}{3}\\
&= \left(3+\frac{1}{2}\right)\cdot 2+\left(3+\frac{1}{2}\right)\cdot\frac{2}{3}\\
&= 3\cdot 2+\frac{1}{2}\cdot 2+3\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{2}{3}\\
&= 6+1+2+\frac{1}{3}\\
&= 9\frac{1}{3}
\end{aligned}
$$

Next, we rework the problem by first changing each mixed number into an improper fraction.

$$3\frac{1}{2}\cdot 2\frac{2}{3}=\frac{7}{\cancelto{1}{2}}\cdot\frac{\cancelto{4}{8}}{3}=\frac{28}{3}=9\frac{1}{3}$$

Summary:

1. It is usually easier to leave the terms as mixed numbers when adding or subtracting mixed numbers.
2. But, it is usually easier to change each mixed number to an improper fraction when multiplying mixed numbers.

Properties for Addition and Multiplication of Fractions

Commutative Property for Fraction Addition $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$ *and Multiplication* $\frac{a}{b}\cdot\frac{c}{d}=\frac{c}{d}\cdot\frac{a}{b}$

Associative Property for Fraction Addition $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$

and Multiplication $\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f}=\frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$

Identity Property for Fraction Addition $\frac{a}{b}+0=0+\frac{a}{b}=\frac{a}{b}$ *and Multiplication.* $\frac{a}{b}\cdot 1=1\cdot\frac{a}{b}=\frac{a}{b}$

Inverse Property for Fraction Multiplication $\frac{a}{b}\cdot\frac{b}{a}=\frac{b}{a}\cdot\frac{a}{b}=1$ where a and b are nonzero. The fraction $\frac{b}{a}$ is called the *multiplicative inverse* of $\frac{a}{b}$ (or *reciprocal*) and vice versa.

Distributive Property of Multiplication over Addition of Fractions $\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\cdot\frac{c}{d}\right)+\left(\frac{a}{b}\cdot\frac{e}{f}\right)$.