Session 23 –Fraction Division

Consider the following two basic problems.

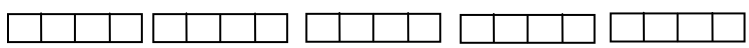
Your mother told you to get $5 in quarters. How many quarters should you bring back?

Dad told you to give the chickens four pails of feed. You are only strong enough to carry half a pail at a time. How many trips do you need to make?

You would probably solve both problems in a similar manner. You would probably solve the first problem by considering that there are four quarters in a dollar. And then since you have $5 and $5 \cdot 4 = 20$, we would have twenty quarters. For the second problem, you would probably consider that there are two halves in each pail so that two trips are needed for one pail. And then since four pails of feed are needed and $4 \cdot 2 = 8$, you would need to make eight trips.

But, if we use the numbers as given in the original problems, the problems are actually division problems based on the repeated-subtraction model for division, since we are asking how many $\frac{1}{4}$ dollars are needed to make $5 ($5 \div \frac{1}{4}$, how many one-fourths are in five), and how many $\frac{1}{2}$ pails are needed to make 4 pails ($4 \div \frac{1}{2}$, how many one-halves are in four). We model the two problems below and write the problem in both the division and multiplication forms.

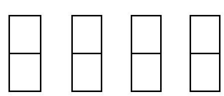
First Problem: We use a fraction strip to represent one dollar.



$$5 \div \frac{1}{4} = 5 \cdot 4 = 20$$

You would bring twenty quarters back.

Second Problem: Again, use a fraction strip to represent one pail.



$$4 \div \frac{1}{2} = 4 \cdot 2 = 8$$

You would need to make eight trips.

Note that both of these basic problems motivate the common rule for division of fractions, "invert the divisor and multiply by the reciprocal". Though most of us solve basic problems like the above two without thinking that we are dividing fractions, we need to understand the connection to division of fractions so that we will be able to solve more complex problems.

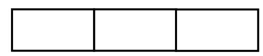
Example: You need to lay tile to create a frieze above a doorway. The tile measures $2\frac{7}{8}$ inches by $2\frac{7}{8}$ inches. If the doorway is $35\frac{1}{2}$ inches wide, how many pieces of tile are needed? *Write a mathematic expression for this problem.*

Solution: $35\frac{1}{2} \div 2\frac{7}{8}$, since we need to find how many $2\frac{7}{8}$ inch pieces fit along $35\frac{1}{2}$ inches.

Model for Division of Fractions

Remember the missing-factor definition for division: $a \div b = c$ if and only if $b \cdot c = a$. With whole numbers this meant that 12 ÷ 3 = 4 precisely because 3 · 4 = 12. Also remember that division answers questions like "how many groups of 3 items can be made out of 12 items?" The division answer above tells us that when we have 12 items we can form 4 groups of 3 items each.

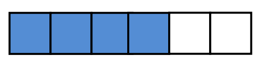
The same relationship defines division of fractions, e.g., $\frac{2}{3} \div \frac{1}{6} = x$ if and only if $\frac{1}{6} \cdot x = \frac{2}{3}$. To see what happens, we solve this problem using a model first. This division problem answers the question "how many groups of $\frac{1}{6}$ can be made out of $\frac{2}{3}$?"

So we begin with a whole divided into 3 equal pieces. 

Since we start with $\frac{2}{3}$, we need to shade 2 of the pieces to represent this amount.

Next we need to determine how many groups of $\frac{1}{6}$ can be formed from the shaded region.

Notice that we can make the whole into six equal pieces in this picture, $\frac{2}{3} = \frac{4}{6}$.



Hence, the whole is in sixths. The shaded region contains 4 of these sixths.

Therefore, $\frac{2}{3} \div \frac{1}{6} = 4$ because we can see that there are 4 of these sixths in the shaded $\frac{2}{3} = \frac{4}{6}$.

Finding the Standard Algorithm for Dividing Fractions

To illustrate, we consider the division problem $\frac{3}{4} \div \frac{3}{5} = x$. Instead of using a model this time, we solve this fraction division problem using the missing-factor definition of division, fraction multiplication, and properties for solving equations. This strategy will lead us to a general rule for computing division of fractions.

The following is an overview of the strategy we will use. A statement-reason table for this strategy appears on the next page. Read through this overview and then study the statement-reason table and try to follow the reasoning being used. In the end, we generate a simple rule for computing division of fractions, which is based on this strategy.

1. By the missing-factor definition of division, we know that $\frac{3}{4} \div \frac{3}{5} = x$ if and only if $\frac{3}{5} \cdot x = \frac{3}{4}$.

2. Remember when we solved equations like $4x = 12$, we used a property of equality to divide both sides of the equation by 4 to get $1 \cdot x = 3$ or $x = 3$. We used a property of equality to get the x alone on the left side of the equation.

3. Using the same strategy on $\frac{3}{5} \cdot x = \frac{3}{4}$, we need to find a value we can multiply both sides of the equation by so that that value times $\frac{3}{5}$ equals 1. Since $\frac{5}{3} \cdot \frac{3}{5} = \frac{15}{15} = 1$, the value we need to multiply both sides of this equation by is $\frac{5}{3}$.
4. Therefore, $\frac{5}{3} \cdot \frac{3}{5} x = \frac{3}{4} \cdot \frac{5}{3}$ and then $1 \cdot x = \frac{5}{4}$ and finally $x = \frac{5}{4} = 1\frac{1}{4}$.

Statement-Reason Summary of Fraction Division Strategy

Statement	Reason
$\frac{3}{4} \div \frac{3}{5} = x$	Original Problem
$\frac{3}{5} \cdot x = \frac{3}{4}$	Missing-Factor Definition of Division
$\frac{5}{3}\left(\frac{3}{5}x\right) = \frac{5}{3} \cdot \frac{3}{4}$	Multiplication Property of Equality (Strategy: use $\frac{5}{3}$ so that $\frac{5}{3} \cdot \frac{3}{5} = 1$)
$\left(\frac{5}{3} \cdot \frac{3}{5}\right)x = \frac{5}{3} \cdot \frac{3}{4}$	Associative Property of Multiplication
$\left(\frac{\cancel{5}^{1}}{\cancel{3}_{1}} \cdot \frac{\cancel{3}^{1}}{\cancel{5}_{1}}\right)x = \frac{5}{\cancel{3}_{1}} \cdot \frac{\cancel{3}^{1}}{4}$	Simplification of Fractions
$1 \cdot x = \frac{5}{4}$	Inverse Property of Multiplication
$x = \frac{5}{4} = 1\frac{1}{4}$	Identity Property of Multiplication

Note the above problem has $x = \frac{3}{4} \div \frac{3}{5} = \frac{3}{4} \cdot \frac{5}{3} = \frac{5}{4} = 1\frac{1}{4}$, which follows the "invert the divisor and multiply by the reciprocal" rule.

In the previous example, we saw how the process of dividing fractions is based on rewriting the problem in its multiplication form and then solving by multiplying both sides of the equation by a value that will simplify the problem to the form 1 · x = ■ where ■ is a fraction multiplication problem. The key to this process is finding pairs of numbers that when multiplied equal one.

Multiplicative Inverse or Reciprocal

When the product of two numbers is one, they are called *reciprocals* or *multiplicative inverses* of each other. For example, $\frac{2}{7}$ and $\frac{7}{2}$ are reciprocals because $\frac{2}{7}\cdot\frac{7}{2}=\frac{14}{14}=1$. This is the motivation for the following property of fractions.

Inverse Property for Fraction Multiplication $\frac{a}{b}\cdot\frac{b}{a}=\frac{b}{a}\cdot\frac{a}{b}=1$ where a and b are nonzero. The fraction $\frac{b}{a}$ is called the *multiplicative inverse* of $\frac{a}{b}$ (or *reciprocal*) and vice versa.

Notice that a reciprocal (multiplicative inverse) can be formed from any common fraction by exchanging the positions of the numerator and denominator. The reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$, of $\frac{4}{9}$ is $\frac{9}{4}$, and $\frac{21}{40}$ is $\frac{40}{21}$. The reciprocal (multiplicative inverse) for a mixed number is found by first changing the mixed number to an improper fraction. For example, $3\frac{2}{7}=\frac{23}{7}$ so the reciprocal of $3\frac{2}{7}$ is $\frac{7}{23}$.

More Examples: $\frac{3}{4}$ has reciprocal $\frac{4}{3}$ since $\frac{\cancel{3}^{1}}{\cancel{4}_{1}}\cdot\frac{\cancel{4}^{1}}{\cancel{3}_{1}}=\frac{1}{1}=1$.

7 has reciprocal $\frac{1}{7}$ since $7\cdot\frac{1}{7}=\frac{7}{7}=1$.

0 has no reciprocal since there is no x such that $0\cdot x=1$.

$4\frac{2}{3}$ has reciprocal $\frac{3}{14}$ since $4\frac{2}{3}\cdot\frac{3}{14}=\frac{\cancel{14}^{1}}{\cancel{3}_{1}}\cdot\frac{\cancel{3}^{1}}{\cancel{14}_{1}}=\frac{1}{1}=1$.

Important Note. The reciprocal of zero is undefined.

The Standard Algorithm for Dividing Fractions

Now we derive the general rule for dividing fractions.

Statement	Reason
$\frac{a}{b}\div\frac{c}{d}=x$	Original Problem
$\frac{c}{d}\cdot x=\frac{a}{b}$	Missing-Factor Definition of Division

$\frac{d}{c}\left(\frac{c}{d}x\right)=\frac{d}{c}\cdot\frac{a}{b}$	Multiplication Property of Equality
$\left(\frac{d}{c}\cdot\frac{c}{d}\right)x=\frac{d}{c}\cdot\frac{a}{b}$	Associative Property of Multiplication
$\left(\frac{d}{c}\cdot\frac{c}{d}\right)x=\frac{a}{b}\cdot\frac{d}{c}$	Commutative Property of Multiplication
$1\cdot x=\frac{a}{b}\cdot\frac{d}{c}$	Inverse Property of Multiplication
$x=\frac{a}{b}\cdot\frac{d}{c}$	Identity Property of Multiplication

Short-cutting all of the algebra steps shown in the previous table, we can generalize fraction division with the following formula.

Invert-and-Multiply Rule for Dividing Fractions:

For *a*, *b*, *c*, and *d*, whole numbers with *b*, *c*, and *d* not equal to zero, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$.

This may be described in words as "invert the divisor and multiply by the reciprocal." Some describe the rule as "change to divide and multiply by the reciprocal."

Example: $\frac{2}{3}\div\frac{4}{7} = \frac{\cancelto{1}{2}}{3}\cdot\frac{7}{\cancelto{2}{4}} = \frac{7}{6} = 1\frac{1}{6}$

"Invert the divisor and multiply by the reciprocal" can seem like a strange and mysterious rule, but as we have seen, it follows from the definition of division, properties of multiplication, and properties of equality (definition, properties, and strategies for solving equations).