Session 37 – Introduction to Integers

How may we compare the values of the following two accounts?

> *Sam had two accounts, a savings account and a loan. She had a certificate of deposit worth $4,360 and a car loan of $8,290. Which account has the greatest value?*

The answer to this question depends on how we think about the situation. If we are asking which situation involves more dollars, then $8,290 involves a greater dollar amount than $4,360. However, if we are asking which value is worth more to Sam, obviously the certificate of deposit, which is money she "owns" is worth more than the loan, which is money she "owes".

In mathematical terms, Sam's CD has a positive value and the loan has a negative value. In mathematics, when we look at just the number of dollars and not whether the value is positive or negative we are talking about the *absolute values*.

We consider two different models to help us understand the properties of positive and negative numbers. First we consider the number line model and then we consider the chip model (set model).

The Set of Integers using the Number Line Model

Consider the two examples below. On this first number line the value "1" is a distance of one unit from zero to the right. We use a vector of length one pointing to the right to represent positive one, 1.
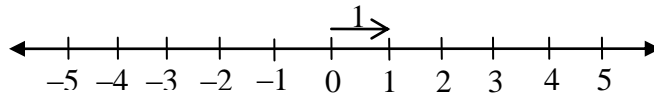
The value −1 is a distance of one unit from zero in the opposite direction. For this reason the value −1 is sometimes referred to as "the opposite of 1". We use a vector of length one pointing to the left to represent negative one, −1.
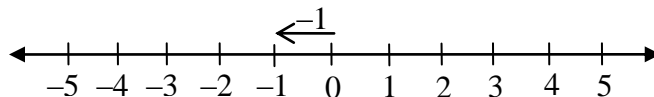
We could write the value positive 1 as +1, but we normally leave out the positive sign. A value with no sign is assumed to be a positive value.

Examples: (a) One step forward, +1.
(b) One step backware, −1.
(c) A temperature of one degree above zero, +1.
(d) A temperature of one degree below zero, −1.

Definition. The set of **integers** is the set {…, −3, −2, −1, 0, 1, 2, 3, …}. Notice that this is the set of whole numbers {0, 1, 2, 3, 4, . . . } together with all of the *opposites*. (The opposite of zero is zero.)

In the number line model, integers represent values as *directed distances*. To find a location on the number line we need to know both the distance (number of units) **and** the direction (whether we should count those units moving left or moving right).

Another term for a directed distance is *vector*. A vector has both distance and direction. In this context an arrow is used to indicate the vector's direction. The arrow used to represent a vector is different from a ray. In geometry, a ray represents an infinite set of points. For a geometric ray, the arrow indicates the direction in which the ray extends indefinitely. A vector is ***not*** a set of points. The length of the vector indicates a specific, ***finite*** distance and the arrow part only indicates the direction. The arrow part of a vector does ***not*** indicate that the distance continues forever.

   Here is one last, important point. Distance by itself is non-directed and is always zero or positive.

   Examples:
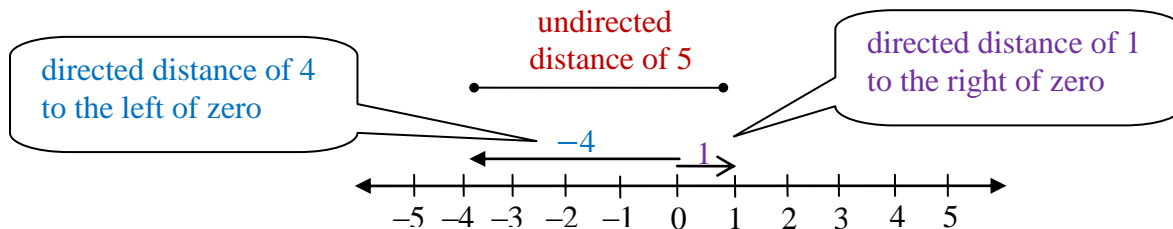
The integer labels on the number line are called the *coordinates* of the points. Since the number line coordinates are measured from zero, zero is called the *origin*.
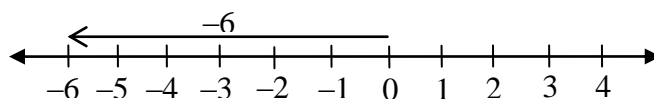
Absolute Value

   Absolute value refers to the distance from zero without reference to direction. Absolute value, since it is a non-directed distance, is always zero or positive. The notation for absolute value is to put straight vertical lines around a value.

   Example: Suppose Pat walked 5 steps forward and Kim walked 6 steps backward. Who took more steps?

   Even though we would represent Pat's steps as 5 and Kim's steps as –6, we would say that Kim took more steps.
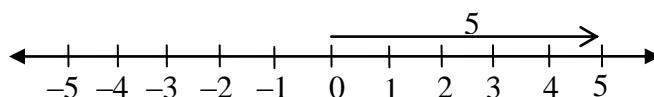
   $|5|$ read as the "absolute value of 5" is 5 since the (non-directed) distance is five units from 0 (the origin).

   $|{-}6|$ read as the "absolute value of –6" is 6 since the (non-directed) distance is six units from 0 (the origin).
   We would write $|5| = 5 < 6 = |{-}6|$. Pat took fewer steps than did Kim.

This problem and the opening problem for this session both motivate the concept of absolute value.

   Example: $|{-}5|$ read as "absolute value of –5" equals 5 because the (non-directed) distance is 5.

The Set of Integers Using the Chip Model (Set Model)

The *chip model* uses colored beans or chips to represent negative integers and positive integers.

A green bean or shaded circle, ●, will be used to represent 1.
A red bean or open circle, ○, will be used to represent –1.

Examples: (a) A deposit of $1 may be represented by ●.
(b) A withdrawl of $1 may be represented by ○.
(c) Receiving 3 candy bars may be represented as ● ● ●.
(d) Giving away 3 candy bars may be represented as ○ ○ ○.

Another important concept with the chip model is that there are many different ways to represent an amount of zero.

Examples:
(a) No chips at all would represent an amount of zero.

(b) ● ○ would represent an amount of zero, e.g., a deposit of $1 and a withdrawal of $1 leaves no change in the balance.

(c) ● ● ● ○ ○ ○ would represent an amount of zero, e.g., receive 3 candy bars and give away 3 candy bars leaves no change in the amount of candy bars.

In other words, two chips of opposite color cancel each other out, so both chips can be removed. This example illustrates the property that $1 + (-1) = -1 + 1 = 0$.
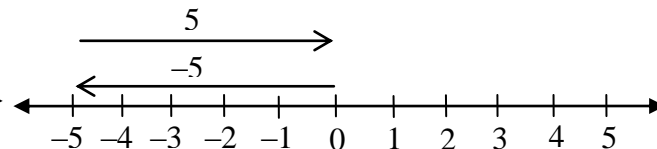
Additive Inverse

We generalize the relationship where opposites of the same absolute value cancel each other out. We further illustrate with examples with both models.

$$3 + (-3) = 0 \qquad\qquad -5 + 5 = 0$$
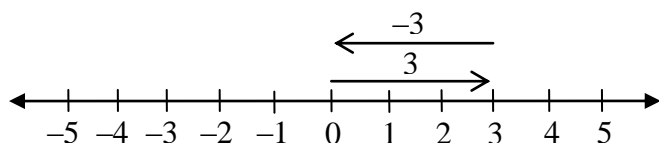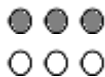
Beginning at zero, in the measurement model, a move of three units to the right followed by three units to the left brings us back to zero.

Beginning at zero, in the measurement model, a move of five units to the left followed by five units to the right brings us back to zero.

We notice that the result of combining a number and its opposite is zero. This motivates the following formal definition.

Additive Inverse: For every integer *n*, there is a unique integer *m* such that $n + m = m + n = 0$. The integer *m* is called the *additive inverse* of *n*. This property of integers is called the *inverse property for integer addition.*

   Examples: Find the additive inverse for each of the following integers.

Integer	7	−5	−13	29	0
Additive Inverse	−7	5	13	−29	0

   If we consider the negative sign as representing the opposite, the above example illustrates that $-(-5) = 5$ and $-(7) = -7$, i.e. the opposite of an integer is equivalent to the additive inverse of the integer. This relationship allows us to give an algebraic definition for absolute value.

Algebraic Definition of Absolute Value: The absolute value of *a* is defined by $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$.

   Examples: (a) $|5| = 5$ since $5 > 0$.

   (b) $|-5| = -(-5) = 5$ since $-5 < 0$.

   (c) $|18| = 18$ since $18 > 0$.

   (d) $|-24| = -(-24) = 24$ since $-24 < 0$.

Comparing the Two Models

The number line is a good model for the expressing positions relative to some specific value, such as graphing on coordinate planes (relative to the origin), temperatures on thermometers (relative to zero degrees or to freezing), and elevation (relative to sea level). It is also good when we need to express fractional parts of the whole, since we can find these on the number line, but it would be difficult to do with beans or chips.

The chip model is a good model for small amounts (with no fractional parts) and for items that have atomic charges such as protons, electrons, and neutrons. It is also a helpful model for understanding how integer arithmetic works.

Extending the Concept of Positive and Negative to Rational Numbers
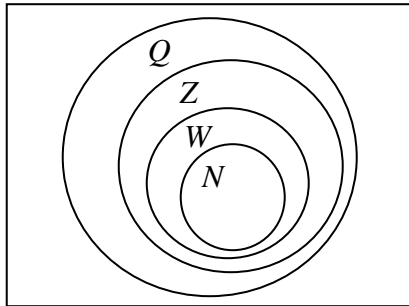
Definition. The set of *rational numbers* is the set $Q = \left\{ \dfrac{a}{b} : a \text{ is an integer and } b \text{ is a natural number} \right\}$.

   Examples: (a)   −5 is a rational number since $-5 = \dfrac{-5}{1}$.

(b) $-2\frac{2}{3}$ is a rational number since $-2\frac{2}{3} = \dfrac{-8}{3}$.

(c) -2.15 is a rational number since $-2.15 = \dfrac{-215}{100} = \dfrac{-43}{20}$.

What, if any, are the relationships between the sets of natural numbers (N), whole numbers (W), integers (Z), and rational numbers (Q)?

The Euler diagram (a variation of a Venn diagram) illustrates that there is a subset relationship between the sets.

$$N \subset W \subset Z \subset Q$$

Note that they are proper subsets. Why?