2.1.3 Analytic Models for Plane Geometry

The ludicrous state of solid geometry made me pass over this branch.
—Plato Republic (429–347 B.C.)

This section lists analytic models that will be used in the study of plane geometry. The models are to help to better understand the axioms for neutral geometry (Euclidean geometry and hyperbolic geometry). As we progress through the SMSG axiom set, a model will be dropped when it no longer satisfies the axioms in use.

Definition. A *distance* function on a set S is a function $d : S \times S \to \Re$ such that for all $A, B \in S$

i. $d(A,B) \geq 0$;
ii. $d(A,B) = 0$ if and only if $A = B$;
iii. $d(A,B) = d(B,A)$.

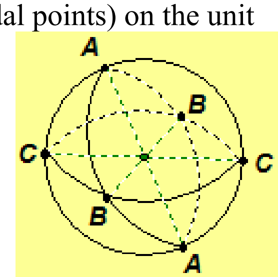
Here, we do not list the Triangle Inequality as part of the definition of a distance function, since other axioms will give us the Triangle Inequality as a theorem.

Discrete Planes. Let S be a nonempty set. The *points* are the elements of S. The *lines* are a specified collection of nonempty subsets of S. The *distance* function $d : S \to [0,\infty)$ is defined by

$$d(P,Q) = \begin{cases} 0 & \text{if } P = Q \\ 1 & \text{if } P \neq Q \end{cases}.$$

Riemann Sphere. The *points* are elements of the unit sphere $S^2 = \{(x,y,z) \in \Re^2 : x^2 + y^2 + z^2 = 1\}$. The *lines* are great circles, G, of the unit sphere S^2 formed by the intersection of S^2 and a plane through the origin; that is, there are $a, b, c \in \Re$ not all zero such that $G = \{(x, y, z) \in S^2 : ax + by + cz = 0\}$. Click here for a java exploration of the Riemann Sphere model.

Modified Riemann Sphere. The *points*, $\{(x, y, z), (-x, -y, -z)\}$, are polar pairs (antipodal points) on the unit sphere $S^2 = \{(x,y,z) \in \Re^2 : x^2 + y^2 + z^2 = 1\}$. The *lines* are the modification of the great circles of the Riemann Sphere. In the illustration of a sphere on the right, the Modified Riemann Sphere has three points A, B, and C. Each polar pair of points (antipodal points) is considered to be a single point. Also, three lines are shown, line AB, line AC, and line BC, which consist of the great circles passing through the polar point pairs. The segments connecting the polar points and the center of the sphere are not in the plane determined by the Modified Riemann Sphere; they are only drawn to help with visualization of the polar pairs. In some sense, the Modified Riemann Sphere may be thought of as a hemisphere with half of the equator.



Cartesian Plane. A *point* is an ordered pair (x, y) where $x \in \Re$ and $y \in \Re$, i.e. $\Re^2 = \{(x,y) : x \in \Re \text{ and } y \in \Re\}$ is the set of all points. The *lines* are either vertical $l_a = \{(x, y) : x = a\}$ or nonvertical $l_{m,b} = \{(x, y) : y = mx + b\}$. Note that these are the definitions for points and lines used in high school algebra.

Euclidean Plane. The *points* and *lines* for the Euclidean plane are the Cartesian points and lines with the addition of a *distance* function $d_E : \Re^2 \to [0,\infty)$ defined by

$$d_E((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The *standard ruler*, $f : l_{m,b} \to \Re$, for a nonvertical line is defined by $f(x,y) = x\sqrt{1+m^2}$.

The *standard ruler*, $f: l_a \to \Re$, for a vertical line is defined by $f(a, y) = y$.

The *angle measure* of $\angle ABC$ is defined by $m(\angle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\|C-B\|}\right)$ where $\langle \cdot, \cdot \rangle$ is the dot product (inner product) of the two vectors and $\|\cdot\|$ is the magnitude of the vector. Note the definition for angle measure is motivated from results in vector calculus or linear algebra. Check your textbooks from those two courses.

Geometer's Sketchpad sketch with tools to explore ruler's in the Euclidean, Taxicab, and Max-Distance planes is available in Appendix B of the Course Title Page - [Prepared Geometer's Sketchpad and GeoGebra Sketches](#).

Taxicab Plane. The *points* and *lines* for the Taxicab plane are the Cartesian points and lines with the addition of a *distance* function $d_T : \Re^2 \to [0, \infty)$ defined by

$$d_T\left((x_1, y_1), (x_2, y_2)\right) = |x_2 - x_1| + |y_2 - y_1|.$$

The *standard ruler*, $f: l_{m,b} \to \Re$, for a nonvertical line is defined by $f(x, y) = (1 + |m|)x$.

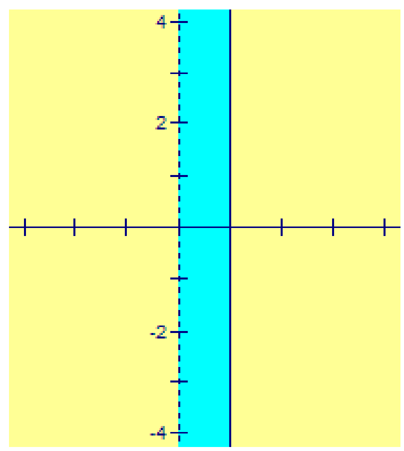
The *standard ruler*, $f: l_a \to \Re$, for a vertical line is defined by $f(a, y) = y$.

Max-Distance Plane. The *points* and *lines* for the Max-distance plane are the Cartesian points and lines with the addition of a *distance* function $d_M : \Re^2 \to [0, \infty)$ defined by

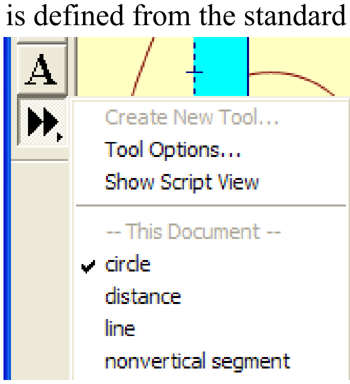
$$d_M\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{|x_2 - x_1|, |y_2 - y_1|\right\}.$$

The *standard ruler*, $f: l_{m,b} \to \Re$, for a nonvertical line is defined by $f(x,y) = \begin{cases} x, & \text{if } |m| \le 1 \\ |m|x, & \text{if } |m| > 1 \end{cases}$.

The *standard ruler*, $f: l_a \to \Re$, for a vertical line is defined by $f(a, y) = y$.



Missing Strip Plane. A *point* is an element of the set $M = \{(x, y) \in \Re^2 : x < 0 \text{ or } x \ge 1\}$. The *lines* are modified Cartesian lines; a line is defined by a Cartesian line intersected with set M (explore Missing Strip lines [GeoGebra html5](#) or [JavaSketchpad](#)), i.e. the lines are elements of the set $L = \{l \cap M : l \text{ is in the Euclidean Plane and } l \cap M \ne \varnothing\}$. The *distance* function $d : M \to [0, \infty)$ for two points on a vertical line is defined by the Euclidean distance between the two points. The *distance* function for two points on a nonvertical line is defined by $d\left((x_1, y_1), (x_2, y_2)\right) = |g_l(x_1, y_1) - g_l(x_2, y_2)|$ where $(x_1, y_1), (x_2, y_2) \in l$ and g_l is defined from the standard ruler f_l of the Euclidean plane by



$$g_l(x,y) = \begin{cases} f_l(x,y), & \text{if } x < 0 \\ f_l(x,y) - \sqrt{1+m^2}, & \text{if } x \ge 1 \end{cases}$$

where m is the slope of line l. Note the Missing Strip plane is simply the Euclidean plane with a vertical strip removed and all the necessary adjustments so that no distance is measured across the missing region. In the diagram on the right, the light blue region (river through a town) is not part of the plane.

A prepared Geometer's Sketchpad sketch and GeoGebra sketch with tools for constructions in the Missing Strip plane are available in Appendix B of the Course Title Page - [Prepared Geometer's Sketchpad and GeoGebra Sketches](#).

Poincaré Half-Plane. The *points* are the elements of the set $H = \{(x, y) \in \Re^2 : y > 0\}$, i.e. the upper half-plane of the Cartesian plane. The *lines* are of two types: vertical rays which are any subset of H of the form ${}_a l = \{(x, y) \in H : x = a\}$, called Type I lines; or semicircles which are any subset of H of the form ${}_c l_r = \{(x, y) \in H : (x-c)^2 + y^2 = r^2\}$, called Type II lines (explore Poincaré lines [GeoGebra html5]() or [JaveSketchpad]()). The *distance* function $d_H : H \to [0, \infty)$ is defined by

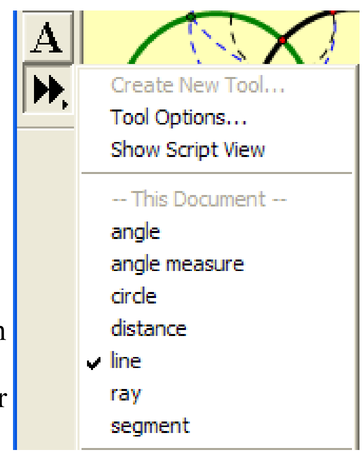
$$d_H\left((x_1, y_1), (x_2, y_2)\right) = \begin{cases} \left|\ln\left(\dfrac{y_2}{y_1}\right)\right| & \text{if } x_1 = x_2 \\ \left|\ln\left(\dfrac{\dfrac{x_2 - c + r}{y_2}}{\dfrac{x_1 - c + r}{y_1}}\right)\right| & \text{if } (x_1, y_1), (x_2, y_2) \in {}_c l_r \end{cases}.$$

The *standard ruler*, $f : {}_a l \to \Re$, for a Type I line is defined by $f(a, y) = \ln y$.

The *standard ruler*, $f : {}_c l_r \to \Re$, for a Type II line is defined by $f(x, y) = \ln\left(\dfrac{x - c + r}{y}\right)$.

Angle measure for $\angle ABC$ is defined by $m(\angle ABC) = \cos^{-1}\left(\dfrac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\|T_{BC}\|}\right)$ where for the ray BA the vector T_{BA} is defined by

$$T_{BA} = \begin{cases} (0, y_A - y_B), & \text{if } \overleftrightarrow{AB} \text{ is a Type I line.} \\ (y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a Type II line, } {}_c l_r, x_B < x_A. \\ -(y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a Type II line, } {}_c l_r, x_B > x_A. \end{cases}$$



Click for a [GeoGebra html5]() or [javasketchpad]() illustration of lines in the Poincaré Half-plane.

Click for a [GeoGebra html5]() or [javasketchpad]() illustration of lines in the Missing Strip plane.

Click for a [GeoGebra html5]() or [javasketchpad]() illustration of angle measure in the Poincaré Half-plane.

A prepared Geometer's Sketchpad sketch and GeoGebra sketch with tools for constructions in the Poincaré Half-plane is available in Appendix B of the Course Title Page - [Prepared Geometer's Sketchpad and GeoGebra Sketches]().
Also, an on-line java based program called [*NonEuclid*]() may be used for constructions in the Poincaré Half-plane at [http://cs.unm.edu/~joel/NonEuclid]().

Another model used for hyperbolic geometry is the Poincaré Disk. A prepared sketch for the Poincaré Disk comes with Geometer's Sketchpad. It is located in the Geometer's Sketchpad program folder: Samples – Sketches – Investigations. Also, [*NonEuclid*]() contains the Poincaré Disk model.

Proposition 2.1. The Euclidean distance function is a distance function.

Proof. Let (x_1, y_1) and (x_2, y_2) be points in the Euclidean plane. We need to show the d_E satisfies the three conditions defining a distance function.

Condition (1). We have $|x_2 - x_1| \geq 0$ and $|y_2 - y_1| \geq 0$. Hence, $d_E\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \geq 0$. Thus condition (1) is satisfied.

Condition (2). $d_E((x_1, y_1), (x_2, y_2)) = 0$ iff $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0$ iff $(x_2 - x_1)^2 + (y_2 - y_1)^2 = 0$ iff $(x_2 - x_1)^2 = 0$ and $(y_2 - y_1)^2 = 0$ iff $x_2 - x_1 = 0$ and $y_2 - y_1 = 0$ iff $x_2 = x_1$ and $y_2 = y_1$ iff $(x_1, y_1) = (x_2, y_2)$. Thus condition (2) is satisfied.

Condition (3).

$$\begin{aligned} d_E((x_1,y_1),(x_2,y_2)) &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(-1)^2(x_1-x_2)^2+(-1)^2(y_1-y_2)^2} \\ &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= d_E((x_2,y_2),(x_1,y_1)). \end{aligned}$$

Thus condition (3) is satisfied.

Hence the Euclidean distance function is a distance function. //

Exercise 2.3. For each model (Euclidean, Taxicab, Max-distance, Missing-Strip, and Poincaré Half-plane), find the distance between points $P(-1, 2)$ and $Q(3, 4)$.

Exercise 2.4. Show the Taxicab distance satisfies the definition of distance.

Exercise 2.5. Show the Max-distance distance satisfies the definition of distance.

Exercise 2.6. Show the Missing Strip distance satisfies the definition of distance.

Exercise 2.7. Show the Poincaré Half-plane distance satisfies the definition of distance.

Exercise 2.8. (a) Show the Hamming distance satisfies the definition of distance. (b) Does the Discrete Plane distance satisfy the definition of distance? Justify.

Exercise 2.9. Define a distance function for the Modified Riemann Sphere.

Exercise 2.10. Sketch and describe a circle for each model.