2.1.2 Historical Overview

Let no one ignorant of geometry enter here. [Inscription above Plato's Academy.]
—Plato (427–347 B.C.)

Mathematics, and more specifically geometry, has a deep history dating back to 1900 B.C. with early work by the Egyptian and Babylonian cultures. Geometry was originally used in practical, everyday applications. Egyptians used geometry in their building of canals and pyramids, and for assessing and collecting taxes. The ancient Babylonians used it for their connection with trade with other cultures. Their mathematics was only improved when their involvement in trade was increased. The word geometry in Greek means "earth measure," or literally, the measure of the earth, which was commonly preformed by Egyptian tax collectors to find the area of land that a subject owned and therefore owed taxes on. The Egyptians had little interest for the theory or reasoning behind their geometry; however, they knew many formulas for determining areas and volumes. The ancient Greeks learned of the study of geometry from the Egyptians. A logical understanding of geometry came through the Greeks study of the Egyptian ideas. The study of geometry, specifically the abstract ideas, theory, and logic behind the mathematics, became more widely studied as Greek mathematicians and philosophers became more common and influential.

Euclid (circa 300 B.C.) was a Greek mathematician for whom Euclidean geometry was named. Little is known about Euclid's life, except for his mathematical accomplishments. His most famous work is the *Elements*, a vast composition which is a consolidation of the then known concepts of mathematics and geometry, including definitions, axioms, and the proofs of over 400 theorems; however, the book did not contain any details or applications of his work. (See Euclid's Elements.) The *Elements* did have its flaws, but they were far overshadowed by its truths and the extraordinary achievements made in the book and its proofs. The book is referred to by many as one of the most influential mathematics books of all time, in part by Euclid's "arrangement of materials, the importance that he placed on the use of minimal set of assumptions, and the natural progression of simple results to the more complex" (Wallace, 5).

In the *Elements*, Euclid noted postulates and axioms that he considered common sense ideas that were assumed to be true and unquestionable. The book starts by listing 23 definitions. Euclid then listed his five postulates that he used as a base to prove the theorems. Throughout the *Elements*, postulates referred to assumptions specific to the study of geometry and axioms were assumptions used throughout the study of mathematics.

Postulates

1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a circle with any center and radius.*
4. *That all right angles equal one another.*
5. *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which*

are the angles less than the two right angles.

Euclid was not satisfied with the fifth postulate because of the difference between it and the other four. This is shown through the fact that the first 28 propositions in his book were proved without the use of the fifth postulate; however, from that point, Euclid went on to use it more extensively.

More recent advances in Euclidean geometry have been made by René Descartes (1596–1650), David Hilbert (1862–1943), and George D. Birkhoff (1884–1944).

Descartes, who is widely known for his work in Aristotelian Philosophy and his famous quote, "I think therefore I am," also worked on new methods of solving Euclidean geometry problems. Descartes laid the stepping stones for modern mathematics, specifically in analytical geometry, through the Cartesian coordinate system. As now commonly used, the Cartesian coordinate system gives every point in a plane two numbers that represent its distance from the horizontal and vertical axes. Descartes' system was more general than the usage of horizontal and vertical axes as we know it today. This new coordinate system helped to show that there was a link between geometry and algebra. He would start with a geometric idea and use the findings to create an algebraic equation to represent it. (Descartes' work in coordinate geometry and the idea of a function, ultimately led to the creation of calculus by Newton and Leibniz.)

David Hilbert, a German mathematician, and George Birkhoff, an American mathematician, also formed different approaches to Euclidean Geometry. "Hilbert was able to construct a rigorous presentation for Euclidean geometry by expanding Euclid's axioms" which caused "the need to provide rather difficult proofs for a number of 'obvious' results" (Wallace, 61). Birkhoff also worked on his own axioms that allowed mathematicians to use numeric values to represent the terms. The metric and synthetic approaches clarified and corrected the flaws in Euclid's system. Both approaches resulted in the same theorems that Euclid had used; however, Birkhoff's approach used a far smaller axiom set than Hilbert and Euclid had used. (See Appendix A–Hilbert's Axioms and Appendix B–Birkhoff's Axioms.)

Proclus Diadochus (412–485) wrote a commentary on the *Elements*. He, like many others, attempted to prove the fifth postulate from the other axioms; however, he and others were unsuccessful. He did however, through his attempt to show that the fifth postulate was not needed, state a new statement that could be used in place of Euclid's Fifth Postulate, since it is equivalent to Euclid's fifth postulate. (It is now known as Playfair's Axiom in honor of John Playfair (1748–1819). Playfair's commentary, written in 1795, suggested replacing Euclid's Fifth Postulate with this axiom.) (See Section 2.7.1 Euclidean Parallel Postulate for more discussion on equivalent statements to Euclid's Fifth Postulate.)

Playfair's Axiom. *Through a point not on a line there is exactly one line parallel to the given line.*

Another failed attempt to prove Euclid's Fifth Postulate came from Girolamo Saccheri (1667–1733) in 1697. Saccheri used a method in which he assumed the fifth postulate to be false in order to prove it from the other postulates by contradiction. Saccheri considered a certain type of quadrilateral, called a *Saccheri quadrilateral*, as a basis for the beginning of his work in attempting to demonstrate the fifth postulate. His work, although it did contain some errors, is considered to be an early attempt at non-Euclidean geometry. (See Section 2.6.2 Saccheri Quadrilaterals for more discussion.)

In the 19th century, Carl Friedrich Gauss (1777–1856), Nikolay Ivanovich Lobachevsky

(1793–1856), and János Bolyai (1802–1860) independently worked on what is now known as non-Euclidean geometry; that is, a geometry in which Euclid's Fifth Postulate is denied. This idea had been attempted by many before, a few of which are listed above; however, they were unable to show the proof correctly.

Gauss, a German mathematician and astronomer, is considered to be the first to recognize the reality of non-Euclidean geometry. Gauss had convinced himself that the fifth postulate was independent of the others. While Gauss was living, it is thought that he feared printing his ideas publicly as it may have had a damaging result on his reputation in mathematics. However, a few of his papers on non-Euclidean geometry were published after his death. (See Section 2.8 Euclidean, Hyperbolic, and Elliptic Geometry for more discussion.)

Bolyai, a Hungarian, and Lobachevsky, a Russian mathematician and educator, were the first to publish any findings on non-Euclidean Geometry. Bolyai and Lobachevsky both worked independently of one another and published very similar findings (the ideas of which are today known as hyperbolic geometry). The papers, at that time, were not highly recognized by the mathematics community.

Around 1860, Georg Friedrich Bernhard Riemann (1826–1866), a German mathematical physicist, showed that it was possible to have a geometry in which no parallel lines exist, working with a geometry that was on the surface of a sphere, now called Elliptical geometry. For his early work in Elliptical geometry, there is a spherical model named in his honor, the Riemann sphere. (See Section 2.7.3 Elliptic Geometry for more discussion.) The names of other theorems and concepts throughout mathematics also honor him and show the importance of his mathematical achievements.

During this time Bolyai and Lobachevsky's non-Euclidean ideas were still not widely accepted nor were they proved valid. Eugenio Beltrami (1835–1900) was one of the first mathematicians to write on Bolyai's and Lobachevsky's ideas. His model was incomplete, but he was able to prove that the first four postulates held and the fifth did not. Beltrami's work was finished by Felix Klein (1849–1925). Klein also did work on other non-Euclidean areas such as Riemann's spherical geometry and helped to distinguish three main areas of non-Euclidean geometry. He also showed how the areas of geometry and abstract algebra are related, which is discussed further in the chapter on transformational geometry.