2.6.1 Parallel Lines without a Parallel Postulate

Mathematics consists of proving the most obvious thing in the least obvious way.
—George Polyá (1887–1985)

Before adding a parallel postulate to our study, we consider several questions about parallel lines. Do parallel lines exist? How can we prove two lines are parallel? What do we know about parallel lines without a parallel postulate?

Definitions.

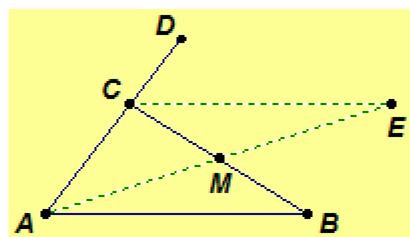
Two lines are *parallel* if and only if they do not intersect.

Given ΔABC., if *A-C-D*, then $\angle BCD$ is an *exterior angle* of ΔABC. Also, $\angle BAC$ and $\angle ABC$ are called *remote interior angles*.

Given line *AB*, line *DE*, and line *BE* such that *A-B-C, D-E-F,* and *G-B-E-H* where *A* and *D* on the same side of line *BE*, then line *BE* is called a *transversal*. Angles $\angle ABE$ and $\angle FEB$ (also $\angle CBE$ and $\angle DEB$) are called *alternate interior angles*.

The next theorem will be useful in proving two lines are parallel. From your high school geometry course, you may remember the theorem: The measure of an exterior angle of a triangle is equal to the sum of the measures of its remote interior angles. That result follows from a Euclidean parallel postulate; however, the following theorem holds in any neutral geometry.

Theorem 2.11. (Exterior Angle Theorem) Any exterior angle of a triangle is greater in measure than either of its remote interior angles.



Outline of the proof. Let ΔABC be given. Let *D* be a point such that *A-C-D,* i.e. $\angle BCD$ is an exterior angle of ΔABC.

1. Let *M* be the midpoint of segment *BC*.
2. *B-M-C* and $\overline{BM} \cong \overline{MC}$.
3. There is a point *E* on ray *AM* such that *A-M-E* and *ME* = *MA*.
4. $\overline{ME} \cong \overline{MA}$.
5. $\angle AMB$ and $\angle EMC$ are vertical angles.
6. $\angle AMB \cong \angle EMC$.
7. $\Delta AMB \cong \Delta EMC$.
8. $\angle ABC = \angle ABM \cong \angle ECM = \angle ECB$.
9. $m(\angle ABC) = m(\angle ECB)$.
10. *E* and *D* are on the same side of line *BC*.
11. *B, M, E* are on the same side of line *CD.*
12. $E \in \text{int}(\angle BCD)$.
13. $m(\angle BCE) + m(\angle ECD) = m(\angle BCD)$.
14. $m(\angle BCD) > m(\angle BCE) = m(\angle ABC)$.
15. The proof of the case for the other remote interior angle is similar.//

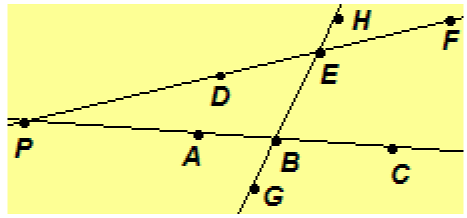
The Exterior Angle Theorem is used in the proof of the Triangle Inequality.

Theorem 2.12. Given a line and a point not on the line, there exists a unique line perpendicular to the given line through the given point.

Theorem 2.13. Two lines perpendicular to the same line are parallel.

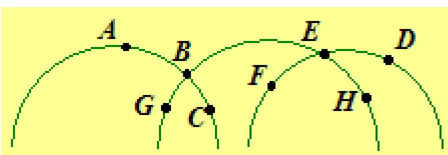
Theorem 2.14. There exist at least two lines that are parallel to each other.

Theorem 2.15. If there is a transversal to two distinct lines with alternate interior angles congruent, then the two lines are parallel.



Outline of the proof. Assume line *AB*, line *DE*, and line *BE* are distinct with *A-B-C, D-E-F,* and *G-B-E-H*. Further, assume *A* and *D* are on the same side of line *BE*. We will prove the contrapositive; therefore, assume line *AB* and line *DE* are not parallel.

1. There exists a point *P* at the intersection of line *AB* and line *DE*.
2. Without loss of generality, assume *P* is on the same side of line *BE* as is *A*.
3. Consider ΔPBE.
4. $\angle BEF$ and $\angle CBE$ are exterior angles of ΔPBE.
5. $m(\angle PBE) < m(\angle BEF)$ and $m(\angle PEB) < m(\angle CBE)$, i.e. $m(\angle ABE) < m(\angle BEF)$ and $m(\angle DEB) < m(\angle CBE)$.
6. $\angle ABE \ncong \angle BEF$ and $\angle DEB \ncong \angle CBE$. //



In general for a neutral geometry, the converse of the above theorem is not valid. The converse is valid in a Euclidean geometry, which is discussed after the Euclidean Parallel Postulate. For an example where the converse is false, consider the Poincaré Half-plane and the illustration on the left. Note that line *AB* is parallel to line *DE,* but the alternate interior angles angle *ABE* and angle *BEF* are not congruent. (Angle *ABE* is a right angle and angle *BEF* is an acute angle.)

Exercise 2.55. For the Poincaré Half-plane, find all lines parallel to the given line through the given point.

(a) (2, 1) and $l_1 = \{(x, y) \in H : x = 1\}$. *(Note the notation error in the graphic equation, it should read* $_1l$ = {(*x, y*)... . *The subscript is before the letter.)*

(b) (2, 1) and $_0l_1 = \{(x, y) \in H : x^2 + y^2 = 1\}$.

Exercise 2.56. Justify each step and fill in any gaps in the proof of the Exterior Angle Theorem.

Exercise 2.57. Prove Theorem 2.12. *(Hint. Use SAS Postulate to prove the existence of a perpendicular line and Exterior Angle Theorem to prove uniqueness.)*

Exercise 2.58. Prove Theorem 2.13.

Exercise 2.59. (a) Prove that given a line and a point not on the line, there is a line parallel to the given line and passing through the point not on the line.

(b) Prove Theorem 2.14.

Exercise 2.60. Justify each step and fill in any gaps in the proof of the Theorem 2.15.

Exercise 2.61. State and prove the AAS Theorem for congruent triangles.