

## Exercises for Chapter Two

*He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.*

—  Plato (429–347 B.C.)

**Exercise 2.1.** Identify the error or errors in the [proof that all triangles are isosceles](#).

**Exercise 2.2.** Identify the error or errors in the [proof of the Rusty Compass Theorem](#).

**Exercise 2.3.** For each model (Euclidean, Taxicab, Max-distance, Missing-Strip, and Poincaré Half-plane), find the distance between points  $P(-1, 2)$  and  $Q(3, 4)$ .

**Exercise 2.4.** Show the Taxicab distance satisfies the definition of distance.

**Exercise 2.5.** Show the Max-distance distance satisfies the definition of distance.

**Exercise 2.6.** Show the Missing Strip distance satisfies the definition of distance.

**Exercise 2.7.** Show the Poincaré Half-plane distance satisfies the definition of distance.

**Exercise 2.8.** (a) Show the [Hamming distance](#) satisfies the definition of distance. (b) Does the Discrete Plane distance satisfy the definition of distance? Justify.

**Exercise 2.9.** Define a distance function for the Modified Riemann Sphere.

**Exercise 2.10.** Sketch and describe a circle for each model.

**Exercise 2.11.** Find the axioms from a high school geometry book that correspond to SMSG Postulates 1 and 5.

**Exercise 2.12.** Prove Theorem 2.2.

**Exercise 2.13.** Show the Missing Strip Plane satisfies (a) SMSG Postulate 1 and (b) SMSG Postulate 5.

**Exercise 2.14.** Show the Poincaré Half-plane satisfies (a) SMSG Postulate 1 and (b) SMSG Postulate 5.

**Exercise 2.15.** Does a Discrete plane satisfy (a) SMSG Postulate 1? Justify. (b) SMSG Postulate 5? Justify.

**Exercise 2.16.** Do the Riemann Sphere and Modified Riemann Sphere satisfy (a) SMSG Postulate 1? Justify. (b) SMSG Postulate 5? Justify.

**Exercise 2.17.** Find the axioms from a high school geometry book that correspond to SMSG Postulates 2, 3, and 4.

**Exercise 2.18.** How do the SMSG Postulates 3 and 4 relate to "real-world" applications?

**Exercise 2.19.** For each model (Euclidean, Taxicab, Max-distance, Missing Strip, and Poincaré Half-plane) find a ruler where  $f(P) = 0$  and  $f(Q) > 0$  for (a)  $P(3, 4)$  and  $Q(3, 7)$ ; and (b)  $P(-1, 3)$  and  $Q(1, 2)$ .

**Exercise 2.20.** Complete the proof that the Ruler Placement Postulate is not independent, Theorem 2.5.

**Exercise 2.21.** Show the stated model satisfies SMSG Postulate 3, the Ruler Postulate, for (a) Taxicab Plane; (b) Max-Distance Plane; (c) Missing Strip Plane; and (d) Poincaré Half-Plane.

**Exercise 2.22.** Does the Modified Riemann Sphere satisfy SMSG Postulate 3, the Ruler Postulate? Explain.

**Exercise 2.23.** Explain why collinear is necessary in the definition of betweenness. (*Hint. Look for an example in either the Taxicab or Max-distance plane where the distance condition is satisfied, but the point would not be on the line.*)

**Exercise 2.24.** Prove a segment has a unique midpoint.

**Exercise 2.25.** Find the midpoint of the segment  $AB$  for each model (Euclidean, Taxicab, Max-distance, Missing Strip, and Poincaré Half-Plane) where (a)  $A(1, 1)$  and  $B(1, 5)$ ; and (b)  $A(-1, 1)$  and  $B(3, 2)$ . (*Show the work using the standard ruler for each model.*)

**Exercise 2.26.** Find the ray  $AB$  for each model (Cartesian, Missing Strip, and Poincaré Half-Plane) where (a)  $A(-3, 1)$  and  $B(-3, 7)$ ; and (b)  $A(-1, 5)$  and  $B(3, 1)$ .

**Exercise 2.27.** An equivalence relation,  $\sim$ , is a relation on a set that satisfies each of the following: (i)  $a \sim a$  (reflexive property) (ii) If  $a \sim b$ , then  $b \sim a$ . (symmetric property) (iii) If  $a \sim b$  and  $b \sim c$ , then  $a \sim c$ . (transitive property).

Prove that  $\cong$  is an equivalence relation for the set of all segments.

**Exercise 2.28.** Is the Plane Separation Postulate discussed or mentioned in the high school geometry book you are using as a reference?

**Exercise 2.29.** Is the Plane Separation Postulate independent of the other axioms? Explain.

**Exercise 2.30.** Find an analytic example that shows the Missing Strip plane does not satisfy Pasch's Postulate.

**Exercise 2.31.** Prove Pasch's Postulate.

**Exercise 2.32.** Find the axioms from a high school book that correspond to SMSG Postulates 11, 12, and 13.

**Exercise 2.33.** Find the measures of the three angles determined by the points  $A(1, 1)$ ,  $B(1, 2)$  and  $C(2, 1)$  where the points are in the (a) Euclidean Plane; and (b) Poincaré Half-plane. Also, find the sum of the measures of the angles of the triangles.

**Exercise 2.34.** Find the angle bisector of  $\angle ABC$ , if  $A(0, 5)$ ,  $B(0, 3)$ , and  $C(2, \sqrt{21})$  where the points are in the (a) Euclidean Plane; and (b) Poincaré Half-plane.

**Exercise 2.35.** Given  $B \in \text{int}(\angle AOC)$ ,  $C \in \text{int}(\angle BOD)$ , and  $m(\angle AOB) = m(\angle COD)$ . Prove or disprove  $m(\angle AOC) = m(\angle BOD)$ .

**Exercise 2.36.** Prove or disprove that all right angles are congruent.

**Exercise 2.37.** Prove or disprove that an angle has a unique bisector.

**Exercise 2.38.** (a) Prove that given a line and a point on the line, there is a line perpendicular to the given line and point on the line.

(b) Prove the existence of two lines perpendicular to each other. Prove the existence of two lines perpendicular to each other.

**Exercise 2.39.** Prove Theorem 2.7.

**Exercise 2.40.** Prove congruence of angles is an equivalence relation on the set of all angles.

**Exercise 2.41.** Find the axiom or theorem from a high school book that corresponds to the Supplement Postulate. Is it an axiom or theorem in the high school book? If it is a theorem, how was it proven?

**Exercise 2.42.** Prove or disprove. If two angles are supplementary, then they form a linear pair.

**Exercise 2.43.** Justify each numbered step and fill in any gaps in the following proof that the Supplement Postulate is not independent of the other axioms.

We need to show that given a linear pair of angles that they are supplementary. Assume  $\angle ABC$  and  $\angle CBD$  form a linear pair of angles. We must show that  $m(\angle ABC) + m(\angle CBD) = 180$ . We do this by showing that  $m(\angle ABC) + m(\angle CBD) < 180$  and  $m(\angle ABC) + m(\angle CBD) > 180$  both lead to contradictions.

Case 1. Suppose  $m(\angle ABC) + m(\angle CBD) < 180$ .

1. There is a unique ray  $BE$  with  $E$  and  $C$  on the same side of line  $AB$  and with  $m(\angle ABE) = m(\angle ABC) + m(\angle CBD)$ .
2.  $C \in \text{int}(\angle ABE)$ .
3. Then  $m(\angle ABC) + m(\angle CBE) = m(\angle ABE)$ .
4. Thus  $m(\angle ABC) + m(\angle CBE) = m(\angle ABC) + m(\angle CBD)$  or  $m(\angle CBE) = m(\angle CBD)$ .
5. Also,  $E \in \text{int}(\angle CBD)$ .
6. Then  $m(\angle CBE) + m(\angle EBD) = m(\angle CBD)$ .
7. Thus  $m(\angle CBD) + m(\angle EBD) = m(\angle CBD)$  or  $m(\angle EBD) = 0$ , which is not possible.

Case 2. Suppose  $m(\angle ABC) + m(\angle CBD) > 180$ .

8.  $0 < m(\angle ABC) + m(\angle CBD) - 180 < 180$ .
9. There is a unique ray  $BF$  with  $F$  and  $C$  on the same side of line  $AB$  and with  $m(\angle ABF) = m(\angle ABC) + m(\angle CBD) - 180$ .
10.  $m(\angle ABC) + m(\angle CBD) - 180 < m(\angle ABC)$ .
11.  $F \in \text{int}(\angle ABC)$ .
12. Thus  $m(\angle ABF) + m(\angle FBC) = m(\angle ABC)$ .
13. Hence  $m(\angle ABC) + m(\angle CBD) - 180 + m(\angle FBC) = m(\angle ABC)$  or  $m(\angle FBC) = 180 - m(\angle CBD)$ .
14. Also,  $C \in \text{int}(\angle FBD)$ .

15. Then  $m(\angle FBC) + m(\angle CBD) = m(\angle FBD)$ .

16. Thus  $180 - m(\angle CBD) + m(\angle CBD) = m(\angle FBD)$  or  $m(\angle FBD) = 180$ , which is not possible.

Thus, by Cases 1 and 2, we must have  $m(\angle ABC) + m(\angle CBD) = 180$ .

**Exercise 2.44.** Prove the Vertical Angle Theorem.

**Exercise 2.45.** Prove that two angles supplement to the same angle are congruent.

**Exercise 2.46.** (a) Prove that if two congruent adjacent angles form a linear pair, then they are right angles.

(b) Prove that the four angles formed by two perpendicular lines are right angles.

**Exercise 2.47.** Prove that a segment has a unique perpendicular bisector.

**Exercise 2.48.** Find the axiom from a high school book that corresponds to the SAS Postulate.

**Exercise 2.49.** Show the SAS Postulate is not satisfied by the (a) Taxicab plane; and (b) Max-distance plane. Thus showing independence.

**Exercise 2.50.** Prove or disprove. If quadrilateral  $ABCD$  is such that segment  $BD$  and segment  $AC$  intersect at a point  $M$  and  $M$  is the midpoint of both segment  $BD$  and segment  $AC$ , then segment  $AB$  is congruent to segment  $CD$ .

**Exercise 2.51.** Prove or disprove. If quadrilateral  $ABCD$  has segment  $CD$  congruent to segment  $CB$  and ray  $CA$  is the bisector of  $\angle DCB$ , then segment  $AB$  is congruent to segment  $AD$ .

**Exercise 2.52.** Prove or disprove. If  $\square ABCD$  is a quadrilateral and  $\overline{AB} \cong \overline{AD}$ , then  $\angle D \cong \angle B$ .

**Exercise 2.53.** State and prove each theorem. (a) SSS Theorem; and (b) ASA Theorem.

**Exercise 2.54.** Prove that all points equidistant from two points  $A$  and  $B$  are on the perpendicular bisector of segment  $AB$ .

**Exercise 2.55.** For the Poincaré Half-plane, find all lines parallel to the given line through the given point.

(a)  $(2, 1)$  and  $l_1 = \{(x, y) \in H : x = 1\}$ .

(b)  $(2, 1)$  and  ${}_0l_1 = \{(x, y) \in H : x^2 + y^2 = 1\}$ .

**Exercise 2.56.** Justify each step and fill in any gaps in the proof of the Exterior Angle Theorem.

**Exercise 2.57.** Prove Theorem 2.12. (Hint. Use SAS Postulate to prove the existence of a perpendicular line and the Exterior Angle Theorem to prove uniqueness.)

**Exercise 2.58.** Prove Theorem 2.13.

**Exercise 2.59.** (a) Prove that given a line and a point not on the line, there is a line parallel to the given line and passing through the point not on the line.

(b) Prove Theorem 2.14.

**Exercise 2.60.** Justify each step and fill in any gaps in the proof of the Theorem 2.15.

**Exercise 2.61.** State and prove the AAS Theorem for congruent triangles.

**Exercise 2.62.** Prove Theorem 2.16.

**Exercise 2.63.** Prove Theorem 2.19.

**Exercise 2.64.** Prove Theorem 2.20.

**Exercises 2.65.** Show the Poincaré Half-plane does not satisfy the Euclidean Parallel Postulate. (a) Use dynamic geometry software to construct an example. (b) Find an analytic example.

**Exercises 2.66.** Show the Poincaré Half-plane does not satisfy Euclid's Fifth Postulate. (a) Use dynamic geometry software to construct an example. (b) Find an analytic example.

**Exercise 2.67.** (a) Prove five of the propositions below using the Euclidean Parallel Postulate and Euclid's Fifth Postulate. (*Once one proposition has been proven, you may use that proposition in the proof another.*) (b) Show the Poincaré Half-plane does not satisfy each of the five propositions. (*May use dynamic geometry software to construct an example.*)

**Use dynamic geometry software with the Poincaré Half-plane for the construction investigations (Geometer's Sketchpad or NonEuclid). The script for Geometer's Sketchpad Poincaré Half-plane is in Appendix B of the Title Page and Index. Make sure you use the Poincaré Half-plane construction tools for lines, rays, segments, angles, circles, etc. Do not use the Euclidean tools.**

**Exercise 2.68.** Construct a triangle in the Poincaré Half-plane. (a) Find the sum of the measures of the angles. (b) Find a triangle where the sum of the measures of the angles is less than 5. (c) Find a triangle where the sum of the measures is more than 175.

**Exercise 2.69.** (a) Construct two perpendicular lines in the Poincaré Half-plane, as illustrated on the right. (b) Prove the two lines are perpendicular. (c) Is this construction valid for a neutral geometry (or Euclidean geometry)? Explain.

**Exercise 2.70.** Use the procedure from Exercise 2 for this exercise. (a) Construct a Saccheri quadrilateral Poincaré Half-plane with the base and sides all congruent. (b) May we say that in a hyperbolic world, "Nobody is a 'square.'"? Explain.

**Exercise 2.71.** (a) Construct a quadrilateral with three right angles in the Poincaré Half-plane. (*This called a Lambert quadrilateral after Johann Lambert (1728–1777).*) (b) What is the measure of the fourth angle? (c) Prove or disprove that a Lambert quadrilateral is a parallelogram. (d) Is the result of part (c) valid for a neutral geometry (or Euclidean geometry)?

**Exercise 2.72.** (a) Construct two parallel lines in the Poincaré Half-plane by constructing them perpendicular to the same line. (b) What happens with the distance between the two parallel lines?

**Exercise 2.73.** Investigate the question: In the Poincaré Half-plane, do two parallel lines have a common perpendicular? If they do not always have a common perpendicular, how are the two parallel lines related?

**Exercise 2.74.** Use the models definitions, not Geometer's Sketchpad for this exercise. Given  $A(0, 2)$ ,  $B$

$(0, 1)$ ,  $C\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ , and  $D(1, \sqrt{3})$  in the Poincaré Half-plane. (a) Show quadrilateral  $ABCD$  is a Saccheri quadrilateral. (b) How do the lengths of summit and base compare?

**Exercise 2.75.** In the Riemann Sphere, what properties are true about all lines perpendicular to a given line?

**Exercise 2.76.** How does a Möbius strip relate to the Modified Riemann Sphere?

**Exercise 2.77.** Describe how is it possible to have a triangle with three right angles.

**Exercise 2.78.** Find an upper bound for the sum of the measures of the angles of a triangle in the Riemann Sphere.

**Exercise 2.79.** Use a ball to represent the Riemann Sphere, construct a Saccheri quadrilateral on the ball. Are the summit angles acute, right, or obtuse? Is the length of the summit more or less than the length of the base? (*Remember the sides of the quadrilateral must be segments of great circles.*)

**Exercise 2.80.** Discuss the questions. Is the physical world Euclidean, hyperbolic, or elliptic? How could you prove the answer to the question? Could the physical world be proved to be hyperbolic or elliptic, but not Euclidean?

[Ch. 2 Euclidean/NonEuclidean TOC](#)

[Table of Contents](#)

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