Appendix A - Hilbert's Axioms for Euclidean Geometry

Mathematics is a game played according to certain rules with meaningless marks on paper.
—David Hilbert (1862–1943)

Introductory Note. Hilbert's Axiom set is an example of what is called a *synthetic geometry*. A synthetic geometry has betweenness and congruence as undefined terms, properties of congruence are given in the axioms, does not have axioms for distance and angle measure. However, Axiom V.1 is related to measurement of length by giving a method of comparing two arbitrary segments.

Undefined Terms *point, line, plane, lie, between,* and *congruence.*

(Some web browsers display some characters incorrectly, an angle shows as ∠, congruent shows as ≅, and the Greek characters alpha and beta show as α, β.)

Group I. Axioms of Incidence

I.1. For every two points *A, B,* there exists a line *m* that contains each of the points *A, B*.
I.2. For every two points *A, B,* there is not more than one line *m* that contain each of the points *A,B*.
I.3. There exist at least two points on a line. There exist at least three points that do not lie on a line.
I.4. For any three points *A, B, C* that do not lie on the same line, there exists a plane α that contains each of the points *A, B, C.* For every plane there exists a point which it contains.
I.5. For any three points *A, B, C* that do not lie on the same line, there exists no more than one plane that contains each of the three points *A, B, C*.
I.6. If two points *A, B* of a line *m* lie in a plane α, then every point of *m* lies in the plane α.
I.7. If two planes α, β have a point *A* in common, then they have at least one more point *B* in common.
I.8. There exist at least four points which do not lie in a plane.

Group II. Axioms of Order

II.1. If point *B* lies between points *A* and *C*, then *A, B, C* are three distinct points of a line, and *B* also lies between *C* and *A*.
II.2. For any two distinct points *A* and *C,* there exists at least one point *B* on the line *AC* such that *C* lies between *A* and *B*.
II.3. Of any three points on a line there exists no more than one that lies between the other two.
II.4. Let *A, B, C* be three points that do not lie on a line and let *m* be a line in the plane *ABC* which does not meet any of the points *A, B, C*. If the line *m* passes through a point of the segment *AB*, it also passes through a point of the segment *AC* or segment *BC.*

Group III. Axioms of Congruence

III.1. If *A, B* are two points on a line *m*, and *A'* is a point on the same or on another line *m'* then it is always possible to find a point *B'* on a given side of the line *m'* through *A'* such that the segment *AB* is congruent to the segment *A'B'*. In symbols *AB* ≅ *A'B'*.

III.2. If two segments are congruent to a third one, they are congruent to each other.
III.3. On the line m let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line m' let $A'B'$ and $B'C'$ be two segments which except for B' also have no point in common. In that case, if $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.
III.4. Let $\angle(h,k)$ be an angle in a plane α and m' a line in a plane α' and let a definite side of m' in α' be given. Let h' be a ray on the line m' that emanates from the point O'. Then there exists in the plane α' one and only one ray k' such that the angle $\angle(h,k)$ is congruent to the angle $\angle(h',k')$ and at the same time all interior points of the angle $\angle(h',k')$ lie on the given side of m'. Symbolically $\angle(h, k) \cong \angle(h',k')$. Every angle is congruent to itself.
III.5. If for two triangles ABC and $A'B'C'$ the congruences $AB \cong A'B'$, $AC \cong A'C'$, $\angle BAC \cong \angle B'A'C'$ hold, then the congruence $\angle ABC \cong \angle A'B'C'$ is also satisfied.

Group IV. Axiom of Parallels

IV. Let m be any line and A be a point not on it. Then there is at most one line in the plane, determined by m and A, that passes through A and does not intersect m.

Group V. Axioms of Continuity

V.1. (*Axiom of measure* or *Archimedes' Axiom*) If AB and CD are any segments, then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.
V.2. (*Axiom of line completeness*) An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms I-III, and form V.1 is impossible.

Defined Terms

- Consider two points A and B on a line m. The set of the two points A and B is called a *segment*. The points between A and B are called the points of the segment AB, or are also said to lie *inside* the segment AB.
- Let A, A', O, B be four points of a line m such that O lies between A and B, but not between A and A'. The points A, A' are then said to lie *on the line m on one and the same side of the point O* and the points A, B are said to lie *on the line m on different sides of the point O*. The totality of the points of the line m that lie on the same side of O is called a *ray* emanating from O.
- Let α be a plane and h, k any two distinct rays emanating from O in α and lying on distinct lines. The pair of rays h, k is called an *angle* and is denoted by $\angle(h,k)$ or $\angle(k,h)$.
- Let the ray h lie on the line h' and the ray k on the line k'. The rays h and k together with the point O partition the points of the plane into two regions. All points that lie on the same side of k' as those of h, and also those that lie on the same side of h' as those on k, are said to lie in the *interior* of the angle $\angle(h,k)$.
- If A, B, C are three points which do not lie on the same line, then the system of three segments AB, BC, CA, and their endpoints is called the *triangle* ABC.

Hilbert, David, *Foundations of Geometry (Grundlagen der Geometrie)*, Second English Edition trans. by Unger,L. LaSalle: Open Court Publishing Company, 1971 (1899).

Appendix B - Birkhoff's Axioms Appendix C - SMSG Axioms

Ch. 2 Euclidean/NonEuclidean TOC Table of Contents

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