3.5.2 Reflection of the Analytic Euclidean Plane Model *Printout*

Nature often holds up a mirror so we can see more clearly the ongoing processes of growth, renewal, and transformation in our lives.

—Mary Ann Brussat

The investigations from the last section indicate that a reflection of the Euclidean plane is an indirect isometry. What form does the matrix of an affine reflection of the Euclidean plane have? Let's investigate that question. To simplify the problem, we first consider the problem with the line $h[0, 1, 0]$ (the x-axis) as the axis of reflection. Let $X(x_1, x_2, 1)$ be an arbitrary point in the Euclidean plane with image X' under the reflection R_h.

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23} \\ 1 \end{bmatrix}$$

Since every point on h is invariant and has the form $X(x_1, 0, 1)$, we must have $x_1 = a_{11}x_1 + a_{13}$ and $0 = a_{21}x_1 + a_{23}$. Thus, since $(0, 0, 1)$ is on h, $a_{13} = 0$ and $a_{23} = 0$. And, since $(1, 0, 1)$ is on h, $a_{11} = 1$ and $a_{21} = 0$. Since R_h maps the point $(0, 1, 1)$ to $(0, -1, 1)$, $a_{12} = 0$ and $a_{22} = -1$. Hence,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We need to show that A is the matrix of the reflection R_h. Matrix A maps an arbitrary point $X(x_1, x_2, 1)$ to the point $X'(x_1, -x_2, 1)$. If X is on h, then $x_2 = 0$ and X' is on h. If X is not on h, then the midpoint $(x_1, 0, 1)$ of segment XX' is on h and the line through X and X' is $l\,[1, 0, -x_1]$. Since $0(1) + 1(0) = 0$, $m\angle(h, l) = \frac{\pi}{2}$ by the definition of the measure of an angle between two lines. Therefore, A is the matrix of the reflection R_h. Since $\det(A) = -1$, the reflection R_h is an indirect isometry.

What is the form of the matrix for a reflection R_l other than R_h? We can apply a method similar to the method used with a rotation that was not centered at the origin. If l is parallel to h, use a translation T that translates h to l and define $R_l = T \circ R_h \circ T^{-1}$. If l is not parallel to h, then l and h intersect at some point P, use a rotation $R_{P,\theta}$ that translates h to l and define $R_l = R_{P,\theta} \circ R_h \circ R_{P,\theta}^{-1}$.

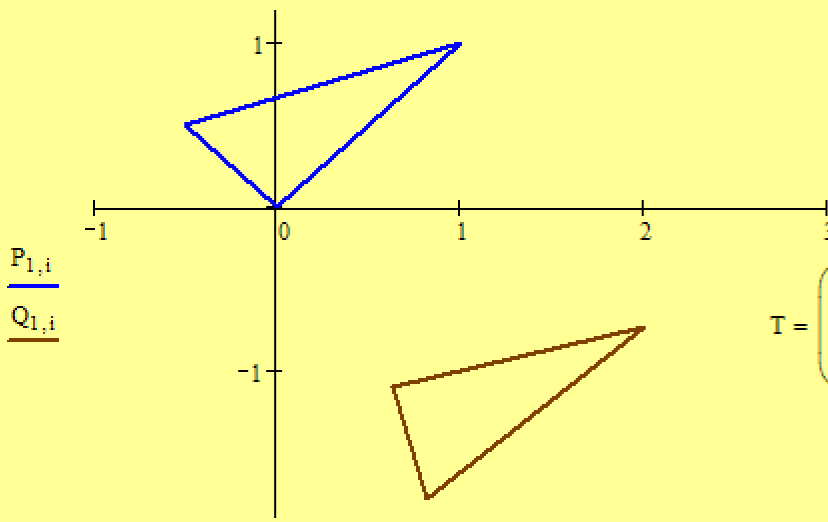
We summarize our results above and from the previous section with the following proposition.

Proposition 3.15. (a) An affine reflection of the Euclidean plane is an indirect isometry. (b) Any affine indirect isometry of the Euclidean plane with exactly one line with all points invariant under the isometry is a reflection. (c) The matrix representation of an affine reflection of the Euclidean plane with axis h*[0, 1, 0] *is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) The matrix representation of an affine reflection of the Euclidean plane with axis l distinct from h is $R_l = T \circ R_h \circ T^{-1}$ where T is a direct isometry that maps line h to line l.

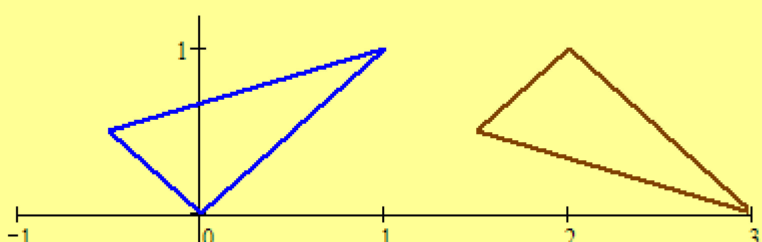
$$T \cdot A \cdot T^{-1} = \begin{pmatrix} 0.5 & 0.866 & 0.634 \\ 0.866 & -0.5 & -1.098 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & -0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 0.634 & 2 & 0.817 & 0.634 \\ -1.098 & -0.732 & -1.781 & -1.098 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0.866 & -0.5 & 3 \\ 0.5 & 0.866 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix TAT^{-1} is the reflection matrix. What is the axis of reflection?
Matrix *P* is the matrix of the three endpoints of the original triangle.
Matrix $Q = TAT^{-1}P$ is the matrix of the reflected endpoints.

$$T \cdot A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & -0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 2 & 3 & 1.5 & 2 \\ 1 & 0 & 0.5 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



Matrix *TA* is the glide reflection matrix. What is the axis?
Matrix *P* is the matrix of the three endpoints of the original triangle.
Matrix *Q* = *TAP* is the matrix of the glide reflected endpoints.

Exercise 3.80. Find a matrix of the reflection R_l where (a) $l[1, -1, 0]$ (b) $l[0, 1, -4]$, then for each reflection find the image of (4, 4, 1) and $\left(\sqrt{2}, 3\sqrt{2}, 1\right)$.

Exercise 3.81. Find a matrix of the reflection R_l where $l\left[1, \sqrt{3}, -\sqrt{3}\right]$, and find the image of (2, 8, 1), (4, 4, 1), and (10, 7, 1).

Exercise 3.82. Find a matrix of the reflection that maps the point $X(3, 8, 1)$ to $Y(5, 1, 1)$ and find the image of $Z(12, 7, 1)$.

Exercise 3.83. Find a matrix of the reflection that maps the line $l[2, 3, -1]$ to $m[2, 3, 5]$.

Exercise 3.84. Find a product of reflections that maps $X(-2, 5, 1)$, $Y(-2, 7, 1)$, and $Z(-5, 5, 1)$ to $X'(4, 3, 1)$, $Y'(6, 3, 1)$, and $Z'(4, 0, 1)$.

Exercise 3.85. Verify part (d) of Proposition 3.15.