# 4.6.4 Alternate Construction of a Projectivity

***Mathematicians create by acts of insight and intuition. Logic then sanctions the conquests of intuition.***
***—Morris Kline (1908–1992)***

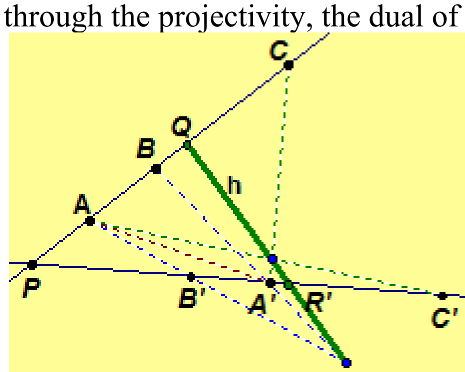
Here, we will formalize the definitions and results from the investigation on the previous page.

***Definition.*** Given two distinct pencils of points with axes $p$ and $p'$, the points $A$, $B$ on $p$ and corresponding points $A'$, $B'$ on $p'$ determine lines $AB'$ and $BA'$ called *cross joins*.

Your exploration of the problem should have led to conjecturing the following theorem known as the *Theorem of Pappus* after Pappus of Alexandria (290-350).

***Theorem 4.15. (Theorem of Pappus) A projectivity between two distinct pencils of points determines a line, which contains the intersection of the cross joins of any two pairs of corresponding points.***

*Proof.* Let $p$ and $p'$ be the axes of two distinct projectively related pencils of points. Assume $P = p \cdot p'$ and $ABC \wedge A'B'C'$ such that $P$ is none of the six points $A, B, C, A', B', C'$. Note the line $AA'$ and the cross joins from $A$ and $A'$ determine two elementary correspondences $(A'A)(A'B)(A'C) \barwedge ABC$ and $A'B'C' \barwedge (AA')(AB')(AC')$. Since $(A'A)(A'B)(A'C) \barwedge ABC$, $ABC \wedge A'B'C'$ and $A'B'C' \barwedge (AA')(AB')(AC')$, we have $(A'A)(A'B)(A'C) \wedge (AA')(AB')(AC')$. Since line $AA'$ maps to itself through the projectivity, the dual of Corollary 4.12 implies that there exists a line $h$ such that this projectivity is a perspectivity with axis $h$, $(A'A)(A'B)(A'C) \overset{h}{\barwedge} (AA')(AB')(AC')$, and does not depend on the choice of $B$ and $C$. Hence, the intersection of the cross joins containing $A$ and $A'$, such as $A'B \cdot AB'$ and $A'C \cdot AC'$, are on $h$.




We need to show the line $h$ is unique and contains the intersections of all pairs of cross joins . Since the above derivation of $h$ used the line $AA'$, we need to show that $h$ is independent of this choice. That is, we need to show that the choice of $BB'$, $CC'$, or some other corresponding pair of points would determine the same line $h$.

Assume $h_x$ is determined from $XX'$ where $X$ and $X'$ are corresponding points from the projectivity that are distinct from $P$. Since $P$ is common to both pencils $p$ and $p'$, let $Q$ be the pre-image of $P$, i.e., $Q' = P$, and let $R'$ be the image of $P$ under the projectivity, i.e., $P = R$. Thus, $X'Q \cdot XQ' = X'Q \cdot XP = Q$ and $X'R \cdot XR' = X'P \cdot XR' = R'$. Hence, $Q$ and $R'$ are on line $h_x$. If $Q$ and $R'$ are distinct, then $h_x = QR'$ for any choice of $XX'$. If $Q$ and $R'$ are not distinct, then $Q = R' = P$. Let $S = X'Y \cdot XY'$ which is on lines determined from $XX'$ and $YY'$. Hence, $h_x = PS = h_y$; that is, the determined line is the same line for any choice of $XX'$ and $YY'$. By the Fundamental Theorem, the image and pre-image of $P$ are unique; therefore, by both cases, $h$ is uniquely determined by $Q$ and $R'$.

The projectivity determines a unique line $h$ that contains the intersection of the cross joins of any two pairs of corresponding points.//

***Definition.*** The line containing the intersection of the cross joins of all pairs of corresponding points is called the *axis of homology*. The point containing the intersection of the cross joins of all pairs of corresponding lines is called the *center of homology*.

***Exercise 4.39.*** Construct the axis of homology for two projectively related pencils of points. Then use the construction to determine an image of an arbitrary point. *(May use dynamic geometry software.)*

***Exercise 4.40.*** Construct the center of homology for two projectively related pencils of lines. Then use the construction to determine an image of an arbitrary line. *(May use dynamic geometry software.)*

4.6.4 Investigation 4.7.1 Conics

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