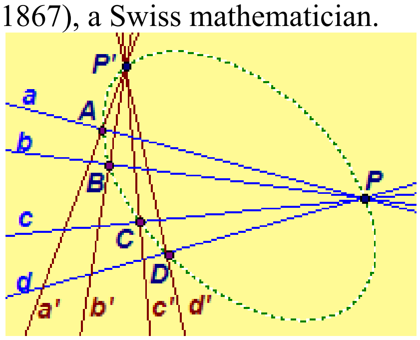
4.7.1 Conics in the Projective Plane

As an achievement it [Apollonius' Conic Sections] is so monumental that it practically closed the subject to later thinkers, at least from the purely geometrical standpoint.
—Morris Kline (1908–1992)

The study of conics first arose from the attempts to solve the problem of duplicating the cube, one of the three famous problems from antiquity (trisecting the angle, duplicating the cube, and squaring the circle.) Two ancient stories are credited with the motivation behind the problem of duplicating the cube. A Greek poet stated that the mythological King Minos desired the tomb of his son, Glaucus, be doubled in size. The error by the poet was stating that this could be accomplished by doubling each dimension. The later story is that the Athenians, in order to end a plague (430 B.C.) which broke out during the war with Sparta, were instructed by the oracle at Delos to double the size of the cubical alter to Apollo.

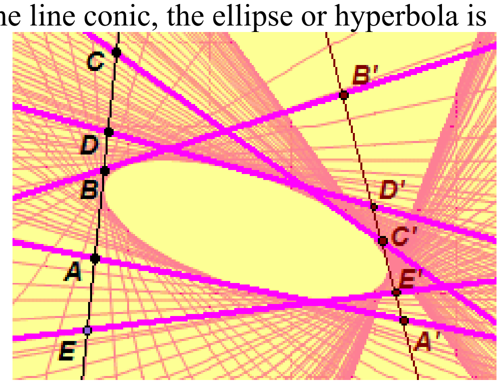
- Menaechmus (375–325 B.C.) is credited with discovering conics in his solution to the problem of doubling the cube.
- Apollonius of Perga (262–190 B.C.) was the first to base the theory of all three conics on sections of one circular cone, right or oblique. He is also credited with giving the names ellipse, parabola, and hyperbola. After Apollonius, the study of conics abated.
- Johannes Kepler (1571–1630) renewed the interest in conics with his work on planetary motion. He showed how the parabola can be considered the limiting case of both the ellipse and hyperbola, where one focus is moved to infinity.
- Blaise Pascal (1623–1662), based on the work of Desargues, wrote an unpublished manuscript on conic sections which included *Pascal's mystic hexagon theorem* (*Pascal's Theorem*). The manuscript is now lost, but was viewed by others, such as Descártes and Leibniz, at the time. The theorem was proved by Pascal in 1640 at the age of 17.

There are many definitions for conics. The definition used here is credited to Jakob Steiner (1796–1867), a Swiss mathematician.



Definition. A *point conic* is the set of points of intersection of corresponding lines of two projectively, but not perspectively, related pencils of lines with distinct centers.

In the diagram on the left, the two pencils of lines are projectively related. Click here for a dynamic illustration of a point conic and a line conic GeoGebra or JavaSketchpad. In the line conic, the ellipse or hyperbola is formed by the envelope of lines determined from the projectively related pencils of points. See the adjacent diagram on the right or investigate further by going to the dynamic illustration of a point conic and line conic GeoGebra or JavaSketchpad.



Definition. A *line conic* is the set of lines that join corresponding points of two projectively, but not perspectively, related pencils of points with distinct axes.

Note in each diagram (or from your investigation) how the centers/axes are related to the point conic/line conic. The centers of the pencil of lines appear to be points of the point conic, and the axes of the pencil of points appear to be lines in the line conic. We state and prove this conjecture with the next theorem.

Theorem 4.16. The centers of the pencils of lines defining a point conic are points of the point conic.

Proof. Let P and P' be the centers of two pencils of lines defining a point conic. Let $p = PP'$. Then p is a line in the pencil with center P. Since the two pencils of lines are projectively related, there is a line p' corresponding to p in the pencil of lines with center P'. Since the pencils of lines are not perspectively related, p and p' are distinct (Dual of Corollary 4.12). Hence, by definition of point conic, $P' = p \cdot p'$ is a point of the point conic.

The argument is the similar for P. Let $p' = PP'$ be a line in the pencil with center P'. Thus, there is a distinct line p corresponding to p' in the pencil of lines with center P. Hence, $P = p \cdot p'$ is a point of the point conic. Therefore, P and P' are both points in the point conic.//

By the dual of the Fundamental Theorem, we know that a projectivity between two pencils of lines is uniquely determined by three pairs of corresponding lines. Thus, the definition of a point conic implies that given any three pairs of corresponding lines a unique projectivity is determined. Further, three points of a point conic are also determined. Putting this observation with Theorem 4.16, we easily obtain five points of a point conic.

A natural question arises: Do any five points, where no three are collinear, determine a point conic? We investigate the question by letting A, B, C, D, E be five distinct points, no three of which are collinear. The lines AD, BD, CD and lines AE, BE, CE are two pencils of lines with centers D and E, respectively. By the dual of the Fundamental Theorem, there is a unique projectivity between the two pencils of lines. Since A, B, and C are intersections of corresponding lines and are noncollinear, the projectivity is not a perspectivity. Hence, by the definition of a point conic and Theorem 4.16, A, B, C, D, E are points of a point conic. We have proven the following theorem.

Theorem 4.17. Any five distinct points, no three collinear, determine a point conic where two of the points are the centers of the respective pencils of lines.

Any two points may be chosen as the centers of the respective pencils. Do different choices for the centers give different point conics? That is, do any five points, no three collinear, determine a unique point conic?

Exercise 4.41. Construct the projectivity as a product of two perspectivities determined by a point conic formed from five points, no three collinear. Use the projectivity to construct a sixth point of the point conic. *(May use dynamic geometry software.)*

Exercise 4.42. Construct the projectivity by using the center of homology determined by a point conic formed from five points, no three collinear. Use the projectivity to construct a sixth point of the point conic. *(May use dynamic geometry software.)*

Exercise 4.43. (a) State the dual of Exercise 4.41 and perform the construction. (b) State the dual of Exercise 4.42 and perform the construction. *(May use dynamic geometry software.)*

Exercise 4.44. Prove that any three distinct points in a point conic are noncollinear.

Exercise 4.45. The definition of point conic includes the phrase "but not perspectively." If this phrase is omitted from the definition, the result would allow all of the points on two lines as points of the point conic. What two lines, defined by the perspectivity, would they be?