# Exercises for Chapter One
# Axiomatic Systems and Finite Geometries

***If you ask mathematicians what they do, you always get the same answer. They think.***
***—M. Egrafov,* Mathematics Magazine *(1992)***

***Exercise 1.1***. Consider the following axiom set.

Postulate 1. There are at least two buildings on campus.
Postulate 2. There is exactly one sidewalk between any two buildings.
Postulate 3. Not all the buildings have the same sidewalk between them.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:
   Theorem 1. There are at least three buildings on campus.
   Theorem 2. There are at least two sidewalks on campus.

c. Show by the use of models that it is possible to have
   exactly two sidewalks and three buildings;
   at least two sidewalks and four buildings; and,
   exactly three sidewalks and three buildings.

d. Is the system complete? Explain.

e. Find two isomorphic models.

f. Demonstrate the independence of the axioms.

***Exercise 1.2.*** Consider the following axiom set.

A1. Every hive is a collection of bees.
A2. Any two distinct hives have one and only one bee in common.
A3. Every bee belongs to two and only two hives.
A4. There are exactly four hives.

a. What are the undefined terms in this axiom set?

b. Deduce the following theorems:
   T1. There are exactly six bees.
   T2. There are exactly three bees in each hive.
   T3. For each bee there is exactly one other bee not in the same hive with it.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

***Exercise 1.3.*** Consider the following axiom set.

P1. Every herd is a collection of cows.

P2. There exist at least two cows.
P3. For any two cows, there exists one and only one herd containing both cows.
P4. For any herd, there exists a cow not in the herd.
P5. For any herd and any cow not in the herd, there exists one and only one other herd containing the cow and not containing any cow that is in the given herd.

a. What are the primitive terms in this axiom set?

b. Deduce the following theorems:
   T1. Every cow is contained in at least two herds.
   T2. There exist at least four distinct cows.
   T3. There exist at least six distinct herds.

c. Find two isomorphic models.

d. Demonstrate the independence of the axioms.

***Exercise 1.4.*** Verify the models satisfy the axioms. *(Since checking every case for Axioms 4 and 5 would be tedious, check enough cases to show sufficient understanding.)* For Axioms 4 and 5, how many cases need to be checked to verify each model?

***Exercise 1.5.*** Show the two models for Fano's geometry are isomorphic.

***Exercise 1.6.*** Show each of the axioms for Fano's geometry is independent.

***Exercise 1.7.*** Write the contradiction argument within the proof of Fano's Theorem 2 to show that *A, B, C, D, E, F,* and *P* are distinct points.

***Exercise 1.8.*** Prove Fano's Theorem 3.

***Exercise 1.9.*** Prove Fano's Theorem 4.

***Exercise 1.10.*** Write the dual for Fano's axioms. Does Fano's geometry satisfy the principle of duality? *(Briefly justify.)*

***Exercise 1.11.*** A four-point geometry.

Undefined terms. *point, line,* and *on*
Axiom 1. There exist exactly four points.
Axiom 2. Two distinct points are on exactly one line.
Axiom 3. Each line is on exactly two points.

a. Show this axiomatic system is consistent.
b. Is the system complete? Explain.
c. Show the axioms are independent.
d. Prove the following theorems.
   T1. If two distinct lines intersect, they intersect in exactly one point.
   T2. A four-point geometry has exactly six lines.
   T3. Each point has exactly three lines on it.
   T4. Each distinct line has exactly one line parallel to it.

e. Show any two models are isomorphic.
f. Write the dual of the four-point geometry creating a four-line geometry.
g. Compare the four-point geometry to Exercise 1.2 of Examples of Axiomatic Systems.

***Exercise 1.12.*** How many cases for each axiom need to be considered to verify the model for a projective plane of order 3?

***Exercise 1.13.*** Write the dual for the axioms of a finite projective plane.

***Exercise 1.14.*** Prove the Dual of Axiom P1.

***Exercise 1.15.*** Prove the Dual of Axiom P2.

***Exercise 1.16.*** Prove the Dual of Axiom P3.

***Exercise 1.17.*** Prove the Dual of Axiom P4.

***Exercise 1.18.*** Identify the axioms for a finite projective plane that are valid in Euclidean geometry. Explain why the others are not valid.

***Exercise 1.19.*** Show the axiomatic system for a finite projective plane is incomplete.

***Exercise 1.20.*** Prove Theorem P4.

***Exercise 1.21.*** How many points and lines are in projective planes of order 4, 13, and 27?

***Exercise 1.22.*** Show the columns of matrix $H_e$ determine a basis for the kernel for matrix $H_p$. *Note all computations are modulo 2.*

***Exercise 1.23.*** Verify that the matrix $H_e$ encodes the sixteen binary numerals 0000, 0001, …, 1111 as given in the Codeword Table. *Note all computations are modulo 2.*

***Exercise 1.24.*** Show the matrix model for Fano's Geometry is isomorphic to the two models given in Section 1.2.

***Exercise 1.25.*** Show the lines in the matrix model for Fano's Geometry as vectors (code words) span the kernel of matrix $H_p$.

***Exercise 1.26.*** Use the generator matrix to encode the words 1011, 0101, and 1001. *Note all computations are modulo 2.*

***Exercise 1.27.*** Assume each codeword has at most a single bit error. Use the parity matrix to identify the position of the error for 1011100, 0101001, and 1001101. *Note all computations are modulo 2.*

***Exercise 1.28.*** Verify that the Hamming distance between each pair of code words in the Codeword Table is at least three.