Exercise 2.2. The Rusty Compass Theorem

It is easier to square the circle than to get round a mathematician.
—Augustus De Morgan (1806–1871)

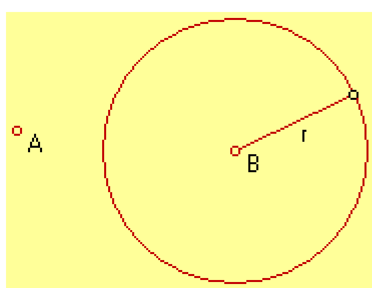
The error may be more difficult to find in this problem than with the previous exercise, since the proof is valid if additional restrictions are made. One of the hardest tasks to do is to find errors in proofs of statements that we know are correct; it is too easy to base proofs on diagrams and preconceptions. This proof is an example of a very common type of error that even good mathematicians make. ***Also, this illustrates how dynamic geometry software may be used to show someone a proof is incorrect when it is difficult to explain the error in logic. (The incorrect proof below was used in a college geometry textbook as the proof for the theorem.)***

The Rusty Compass Theorem (or Compass Equivalence Theorem). Given a circle centered at a point B with radius r. Let A be any point distinct from B. A straightedge and collapsing compass *(Euclidean straightedge and compass)* ***can be used to construct a circle centered at A that is congruent to the given circle centered at B with radius r.***

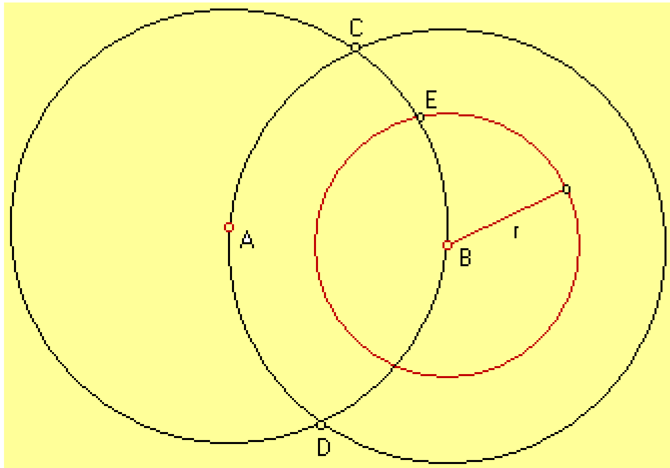
(a) Identify the error/s in the proof. For what cases is the proof valid? For what cases is the proof invalid? Explain.
(b) Identify and write a definition for each term used in the statement of the theorem and proof.
(c) Identify and state any assumptions made in the proof.
(d) Identify and state any theorems used in the proof.
(e) The Compass Equivalence Theorem or Rusty Compass Theorem is valid in Euclidean geometry, though the following proof is incorrect. Is it valid in hyperbolic geometry? Explain.

(Hint. If you are unable to find the errors from reading the proof, use dynamic geometry software - such as Geometer's Sketchpad or GeoGebra - to construct the figure based on the steps in the proof. After making the diagram, drag some of the points to different locations; you should see the proof fall apart.)

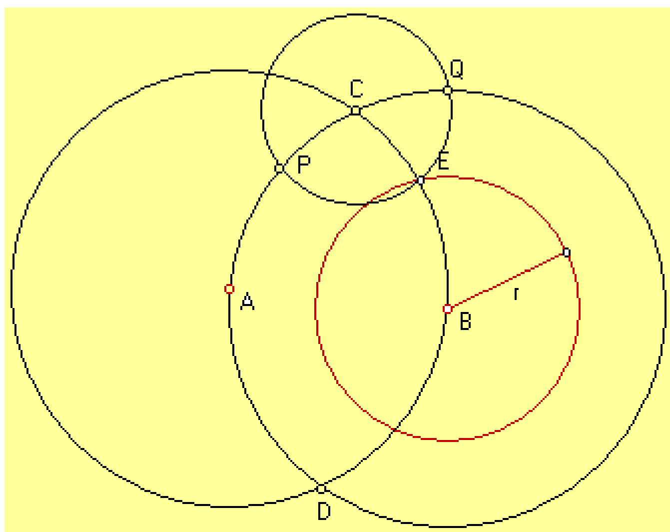
Proof. Consider circle $C(B, r)$ and point A distinct from B.



Our objective is to construct a circle $C(A, r)$ using only a straightedge and a collapsing compass. Construct circles $C(A, AB)$ and $C(B, BA)$ and let C and D be the points of intersection of $C(A, AB)$ and $C(B, BA)$. Let E be one of the points of intersection of $C(B, r)$ and $C(A, AB)$.



Next we construct circle $C(C, CE)$ and let P be a point of intersection of circle $C(C, CE)$ with $C(B, BA)$. We claim that $AP = r$, so that $C(A, AP)$ is the desired circle implied by the statement of the theorem.



To show that $AP = r$, it will suffice to show that $AP = BE = r$, since segment BE and r are radii of the same circle. We will show that $\Delta APC \cong \Delta BEC$ by SAS. Since radii of a circle are congruent, $\overline{CP} \cong \overline{CE}$ and $\overline{AC} \cong \overline{BC}$. Hence, it only remains for us to show that $\angle ACP \cong \angle BCE$.

Since $C(B, BA) \cong C(A, AB)$, $\overline{BP} \cong \overline{AE}$. And since $\overline{CP} \cong \overline{CE}$ and $\overline{AC} \cong \overline{BC}$, we have $\Delta BCP \cong \Delta ACE$ by SSS. Thus, $\angle BCP \cong \angle ACE$. Hence,

$$m(\angle ACP) = m(\angle BCP) - m(\angle BCA) = m(\angle ACE) - m(\angle ACB) = m(\angle BCE).$$

Thus, $\Delta APC \cong \Delta BEC$ which implies that $AP = BE = r$. Therefore, $C(A, AP)$ is congruent to $C(B, r)$.//