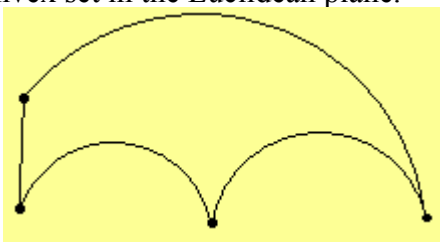
2.4.1 Plane Separation Postulate

Whoever… proves his point and demonstrates the prime truth geometrically should be believed by all the world, for there we are captured.
— Albrecht Dürer (1471–1528)

An important axiom that is often not considered in a high school geometry course is the Plane Separation Axiom. The Missing Strip plane is a model that satisfies all the axioms we have discussed to this point. It is included for the sole purpose of demonstrating the necessity for the Plane Separation Axiom.

Definition. A set S is *convex* if for every two points P and Q in S, the segment $\overline{PQ}$ is a subset of S.

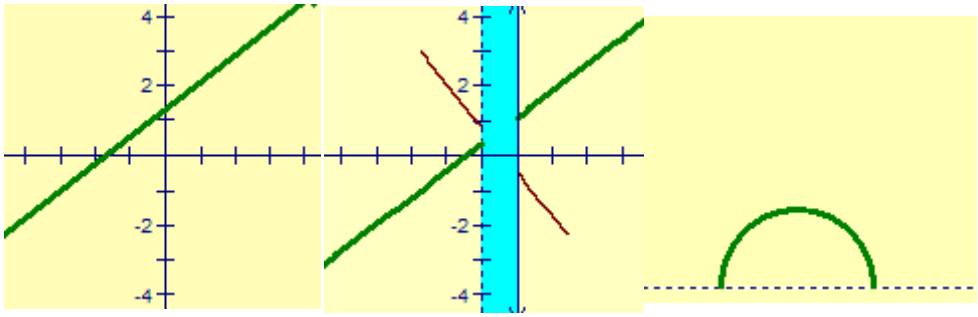
Consider the interiors of the above three figures. Note that the first and last figures have segments that are not completely contained in their interiors. The first and last sets are not convex. It is important to note that the definition of convex depends on a segment; a set that is convex in the Poincaré Half-plane may not be convex as a set in the Euclidean plane. Consider the interior of the following quadrilateral constructed in the Poincaré Half-plane. The set is convex in the Poincaré Half-plane but is not a convex set in the Euclidean plane.



Postulate 9. (*Plane Separation Postulate*) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that: (i) each of the sets is convex; and (ii) if P is in one set and Q is in the other, then segment $\overline{PQ}$ intersects the line.

Definitions. Each of the two convex sets is called a *half-plane,* and the line is called the *edge*. Points in the same half-plane are said to be on the *same side*. Two points in different half-planes determined by a line are said to be on *opposite sides*.

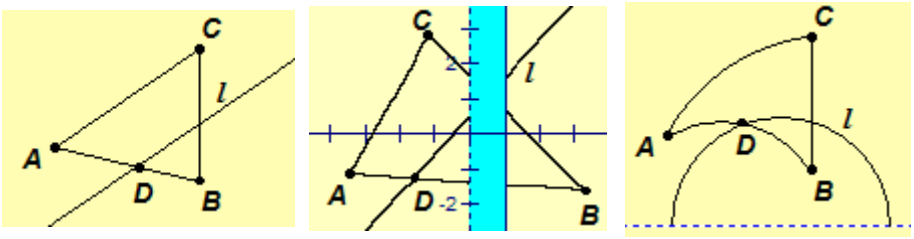
Note that the line and the two half-planes are disjoint sets; that is, a line forms three disjoint sets, the line and two half-planes. To illustrate the Plane Separation Postulate, consider the Cartesian plane, Missing Strip plane, and Poincaré Half-plane.



The line in the Cartesian plane separates the plane into two convex sets, the region above the line and the region below the line. The line in the Missing Strip plane does ***not*** separate the plane into two convex sets. See the illustration, the segment shown does not intersect the line since the light blue region between 0 and 1 does not exist in the plane. The line in the Poincaré Half-plane separates the plane into two convex sets. *(Remember the shape of a line and segment in the Poincaré Half-plane.)*

A statement that is sometimes used in place of the Plane Separation Postulate is Pasch's Postulate. It can be proven that the Plane Separation Postulate and Pasch's Postulate are equivalent, Moritz Pasch (1843-1930). Since this is a survey course, we will not prove their equivalency here. (The proofs may be found in books on the foundations of geometry such as *Geometry: A Metric Approach with Models* by Millman and Parker.)

Theorem 2.6 (Pasch's Postulate). For any line l, any triangle ΔABC, and any point D on l such that D is between A and B, then either l intersects segment AC or l intersects segment BC.



The above diagrams illustrate one example of Pasch's Postulate in the Euclidean plane, Missing Strip plane, and Poincaré Half-plane. Is Pasch's Postulate satisfied in each plane?

Reminder. Prepared Geometer's Sketchpad sketches and GeoGebra sketches with tools for constructions in the Missing Strip plane and Poincaré Half-plane are available in Appendix B of the Course Title Page - Geometer's Sketchpad and GeoGebra Prepared Sketches and Scripts. Also, an on-line java based program called *NonEuclid* may be used for constructions in the Poincaré Half-plane at http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html.

Exercise 2.28. Is the Plane Separation Postulate discussed or mentioned in the high school geometry book you are using as a reference?

Exercise 2.29. Is the Plane Separation Postulate independent of the other axioms? Explain.

Exercise 2.30. Find an analytic example that shows the Missing Strip plane does not satisfy Pasch's Postulate.

Exercise 2.31. Prove Pasch's Postulate.

2.3 Distance and Ruler Axioms 2.4.2 Angles and Angle Measure