2.5.1 Supplement Postulate

A work of morality, politics, and criticism will be more elegant, other things being equal, if it is shaped by the hand of geometry.

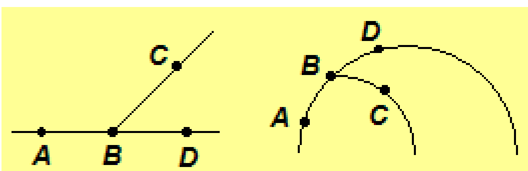
—Bernard le Bovier de Fontenelle (1657–1757)

Definitions.

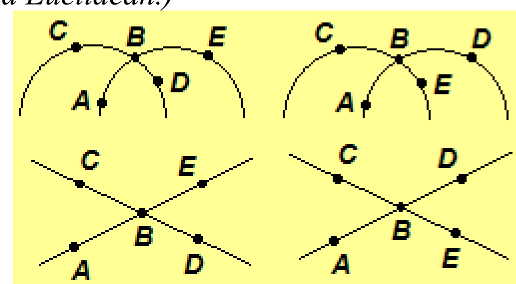
Two distinct angles are said to be *supplementary angles* if the sum of their measures is 180.

Two angles $\angle ABC$ and $\angle CBD$ are a *linear pair* if *B* is between *A* and *D*. *(Two left-hand figures below – Euclidean and Poincaré Half-plane.)*

Two angles $\angle ABC$ and $\angle DBE$ are *vertical angles* if either *A-B-E* and *C-B-D*, or *A-B-D* and *C-B-E*. *(Four right-hand figures below – Poincaré Half-plane and Euclidean.)*



Linear Pair



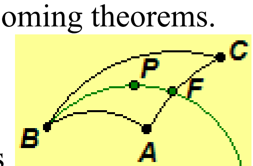
Vertical Angles

Postulate 14. (*Supplement Postulate*) If two angles form a linear pair, then they are supplementary.

The Supplement Postulate is not independent of the other axioms. Since the proof does not add insight into better understanding and is not simple, the statement is taken as an axiom instead of a theorem for most high school geometry courses. The proof that the Supplement Postulate is not independent is given below; Exercise 2.43 asks the reader to provide the reasons for each step of the proof.

Theorem 2.8 (Vertical Angle Theorem) Vertical angles are congruent.

The next theorem, called the Crossbar Theorem, is needed to prove some of the coming theorems. The question the Crossbar Theorem addresses is: How do we know that a ray with one endpoint at a vertex of a triangle that contains an interior point of a triangle intersects the opposite side of the triangle? The answer seems obvious when using Euclidean geometry, but that is making an assumption. The result seems less obvious in the Poincaré Half-plane. Could the ray curl so that it intersects a side that contains the vertex? Or, is there a model where the ray curls in such a way that it never leaves the interior of the triangle? Again, since this is a survey course, we state the Crossbar Theorem here without proof. The theorem follows from the Plane Separation Postulate and Pasch's Postulate.



Theorem 2.9. (Crossbar Theorem) If P is an interior point of angle ABC, then ray BP and segment AC intersect in a unique point F and A-F-C.

Exercise 2.41. Find the axiom or theorem from a high school book that corresponds to the Supplement Postulate. Is it an axiom or theorem in the high school book? If it is a theorem, how was it proven?

Exercise 2.42. Prove or disprove. If two angles are supplementary, then they form a linear pair.

Exercise 2.43. Justify each numbered step and fill in any gaps in the following proof that the Supplement Postulate is not independent of the other axioms.

We need to show that given a linear pair of angles that they are supplementary. Assume $\angle ABC$ and $\angle CBD$ form a linear pair of angles. We must show that $m(\angle ABC)+m(\angle CBD)=180$. We do this by showing that $m(\angle ABC)+m(\angle CBD)<180$ and $m(\angle ABC)+m(\angle CBD)>180$ both lead to contradictions.

Case 1. Suppose $m(\angle ABC)+m(\angle CBD)<180$.

1. There is a unique ray BE with E and C on the same side of line AB and with $m(\angle ABE)=m(\angle ABC)+m(\angle CBD)$.
2. $C\in \text{int}(\angle ABE)$.
3. Then $m(\angle ABC)+m(\angle CBE)=m(\angle ABE)$.
4. Thus $m(\angle ABC)+m(\angle CBE)=m(\angle ABC)+m(\angle CBD)$ or $m(\angle CBE)=m(\angle CBD)$.
5. Also, $E\in \text{int}(\angle CBD)$.
6. Then $m(\angle CBE)+m(\angle EBD)=m(\angle CBD)$.
7. Thus $m(\angle CBD)+m(\angle EBD)=m(\angle CBD)$ or $m(\angle EBD)=0$, which is not possible.

Case 2. Suppose $m(\angle ABC)+m(\angle CBD)>180$.

8. $0<m(\angle ABC)+m(\angle CBD)-180<180$.
9. There is a unique ray BF with F and C on the same side of line AB and with $m(\angle ABF)=m(\angle ABC)+m(\angle CBD)-180$.
10. $m(\angle ABC)+m(\angle CBD)-180<m(\angle ABC)$.
11. $F\in \text{int}(\angle ABC)$.
12. Thus $m(\angle ABF)+m(\angle FBC)=m(\angle ABC)$.
13. Hence $m(\angle ABC)+m(\angle CBD)-180+m(\angle FBC)=m(\angle ABC)$ or $m(\angle FBC)=180-m(\angle CBD)$.
14. Also, $C\in \text{int}(\angle FBD)$.
15. Then $m(\angle FBC)+m(\angle CBD)=m(\angle FBD)$.
16. Thus $180-m(\angle CBD)+m(\angle CBD)=m(\angle FBD)$ or $m(\angle FBD)=180$, which is not possible.

Thus, by Cases 1 and 2, we must have $m(\angle ABC)+m(\angle CBD)=180$.

Exercise 2.44. Prove the Vertical Angle Theorem.

Exercise 2.45. Prove that two angles supplement to the same angle are congruent.

Exercise 2.46. (a) Prove that if two congruent adjacent angles form a linear pair, then they are right angles.

(b) Prove that the four angles formed by two perpendicular lines are right angles.

Exercise 2.47. Prove that a segment has a unique perpendicular bisector.

2.4.2 Angles and Angle Measure ⇔ 2.5.2 SAS Postulate

Ch. 2 Euclidean/NonEuclidean TOC | Table of Contents