# 2.7.1 Euclidean Parallel Postulate

***This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties.***
***—Proclus (410–485)***

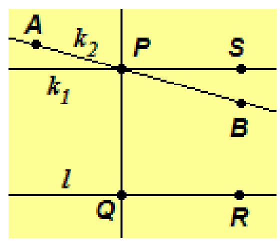
***Euclid's Fifth Postulate.*** That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

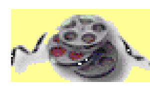
***SMSG Postulate 16.*** (*Euclidean Parallel Postulate*) Through a given external point there is at most one line parallel to a given line.

***Playfair's Axiom.*** Through a point not on a line there is exactly one line parallel to the given line.

Playfair's Axiom is named after John Playfair (1748–1819), a Scottish physicist and mathematician, though many others had used it much earlier. Since we have shown the existence of a parallel line, it is clear that SMSG Postulate 16 (Euclidean Parallel Postulate) and Playfair's Axiom are equivalent. Further, Euclid's Fifth Postulate and the Euclidean Parallel Postulate are equivalent.

***Theorem 2.21. In a neutral geometry, Euclid's Fifth Postulate is equivalent to the Euclidean Parallel Postulate.***



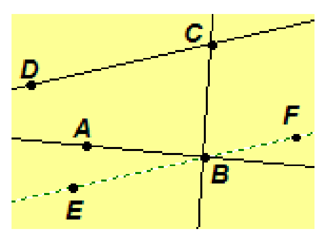



*Proof.* First use Euclid's Fifth Postulate to prove the Euclidean Parallel Postulate. Let $l$ be a line and $P$ be a point not on $l$. By Theorem 2.12, there is a unique line perpendicular to a given line through a given point; therefore, there is a point $Q$ on $l$ such that line $PQ$ is perpendicular to $l$. Also, there is a unique line $k_1$ through $P$ such that line $PQ$ is perpendicular to $k_1$. By Theorem 2.13, two lines perpendicular to the same line are parallel; therefore, $k_1$ is parallel to $l$ and $P$ is on $k_1$.

We need to show that $k_1$ is the unique line parallel to $l$ through $P$. Let $k_2$ be another line through $P$ such that $k_1$ and $k_2$ are distinct lines. Let $A$ and $B$ be distinct points on $k_2$ such that $A$-$P$-$B$ and $B$ and $Q$ are on the same side of $k_1$. Let $R$ be on $l$ and $S$ be on $k_1$ such that $B$, $R$, and $S$ are all on the same side of line $PQ$. Hence, since $B$ and $S$ are on the same side of line $PQ$ and $B$ and $Q$ are on the same side of line $PS$, by the definition of the interior of an angle, $B \in \operatorname{int}(\angle QPS)$. Since line $PQ$ is perpendicular to both $l$ and $k_1$, $\angle QPS$ and $\angle RQP$ are right angles; i.e. $m(\angle QPS) = m(\angle RQP) = 90$. Since $B \in \operatorname{int}(\angle QPS)$, $m(\angle QPB) < m(\angle QPS) = 90$. Hence, $m(\angle RQP) + m(\angle QPB) < m(\angle RQP) + m(\angle QPS) = 180$. Therefore, since $B$ and $R$ are on the same side of line $PQ$ and $m(\angle RQP) + m(\angle QPB) < 180$, by Euclid's Fifth Postulate, $k_2 = \overleftrightarrow{PB}$ and $l = \overleftrightarrow{QR}$ intersect on the same side as $B$ and $R$. Hence, $k_1$ is the unique line parallel to $l$ that contains $P$.

Next, use the Euclidean Parallel Postulate to prove Euclid's Fifth Postulate. Given a line $BC$ and two points $A$ and $D$ on the same side of line $BC$ with

$$m(\angle ABC) + m(\angle BCD) < 180. \quad (1)$$




We need to show that ray $BA$ intersects ray $CD$. By the Angle Construction Postulate, there is a ray $BE$ with $E$ and $A$ on the same side of line $BC$ such that

$$m(\angle CBE) = 180 - m(\angle BCD). \quad (2)$$

Let $F$ be a point such that $E$-$B$-$F$, then $\angle EBC$ and $\angle CBF$ are a linear pair. Hence $\angle EBC$ and $\angle CBF$ are supplementary. Hence,

$$m(\angle EBC) + m(\angle CBF) = 180. \quad (3)$$

By (2) and (3), $m(\angle BCD) = 180 - m(\angle CBE) = 180 - m(\angle EBC) = m(\angle CBF)$. Hence $\angle BCD \cong \angle CBF$. Since *D* and *F* are on opposite sides of line *BC*, $\angle BCD$ and $\angle CBF$ are alternate interior angles. Hence by Theorem 2.15, line *EB* is parallel to line *DC*. By (1) and (2), $m(\angle ABC) < 180 - m(\angle BCD) = m(\angle CBE)$. Hence line *AB* and line *EB* are distinct lines through *B*. Thus, by the Euclidean Parallel Postulate, line *AB* is not parallel to line *DC*.

By Theorem 2.7, since $m(\angle ABC) < m(\angle EBC)$, we have $A \in \text{int}(\angle EBC)$. Thus, since *A* and *C* are on the same side of line *EB*, $\overrightarrow{BA} \setminus \{B\}$ and *C* are on the same side of line *EB*. Since line *EB* and line *DC* are parallel, line *DC* is on one side of line *EB*. Hence ray *BA* intersects line *CD*.

Since *A* and *D* are on the same side of line *BC*, $\overrightarrow{BA} \setminus \{B\}$ and $\overrightarrow{CD} \setminus \{C\}$ are on the same side of line *BC*. Hence, ray *BA* intersects ray *CD*.//

There are many statements that are equivalent to the Euclidean Parallel Postulate, which could be used as the axiom. We list several of them below after the exercises. How many of them can you show are equivalent? The exercises ask you to prove one direction on a few of the statements and to find a counter-example in the Poincaré Half-plane.

***Exercises 2.65.*** Show the Poincaré Half-plane does not satisfy the Euclidean Parallel Postulate. (a) Use dynamic geometry software to construct an example. (b) Find an analytic example.

***Exercises 2.66.*** Show the Poincaré Half-plane does not satisfy Euclid's Fifth Postulate. (a) Use dynamic geometry software to construct an example. (b) Find an analytic example.

***Exercise 2.67.*** (a) Prove five of the propositions below using the Euclidean Parallel Postulate and Euclid's Fifth Postulate. *(Once one proposition has been proven, you may use that proposition in the proof of another.)* (b) Show the Poincaré Half-plane does not satisfy each of the five propositions. *(May use dynamic geometry software to construct an example.)*

***Euclidean Proposition 2.1.*** There exists a line and a point not on the line such that there is a unique line through the point that is parallel to the line.

Note how this proposition differs from Playfair's Axiom. This proposition only says that at least one such point and line exist; whereas, Playfair's Axiom says that it is true for every line and point not on the the line. The surprising result that in a neutral geometry this proposition implies Playfair's Axiom is called the *All or None Theorem*. The result is surprising since we only need existence of the parallel property for one line and one point not on the line to know that the parallel property is true everywhere. A proof of the All or None Theorem may be found in either *Elementary Geometry from an Advanced Standpoint* by Moise, or *Geometry: A Metric Approach with Models* by Millman and Parker.

***Euclidean Proposition 2.2.*** If *A* and *D* are points on the same side of line *BC* and line *BA* is parallel to line *CD*, then $m(\angle ABC) + m(\angle BCD) = 180$.

***Euclidean Proposition 2.3.*** If $l_1$, $l_2$, $l_3$ are three distinct lines such that $l_1$ is parallel to $l_2$ and $l_2$ is parallel to $l_3$, then $l_1$ is parallel to $l_3$.

***Euclidean Proposition 2.4.*** If $l_1$, $l_2$, $l_3$ are three distinct lines such that $l_1$ intersects $l_2$ and $l_2$ is parallel to $l_3$, then $l_1$ intersects $l_3$.

***Euclidean Proposition 2.5.*** A line perpendicular to one of two parallel lines is perpendicular to the other.

***Euclidean Proposition 2.6.*** If $l_1$, $l_2$, $l_3$, $l_4$ are four distinct lines such that $l_1$ is parallel to $l_2$, $l_3$ is perpendicular to $l_1$, and $l_4$ is perpendicular to $l_2$, then $l_3$ is parallel to $l_4$.

***Euclidean Proposition 2.7.*** Every two parallel lines have a common perpendicular.

***Euclidean Proposition 2.8.*** The perpendicular bisectors of the sides of a triangle intersect at a point.

***Euclidean Proposition 2.9.*** There exists a circle passing through any three noncollinear points.

***Euclidean Proposition 2.10.*** There exists a point equidistant from any three noncollinear points.

***Euclidean Proposition 2.11.*** A line intersecting and perpendicular to one side of an acute angle intersects the other side.

***Euclidean Proposition 2.12.*** Through any point in the interior of an angle there exists a line intersecting both sides of the angle not at the vertex.

***Euclidean Proposition 2.13.*** If two parallel lines are cut by a transversal, then the alternate interior angles are congruent. *(The converse of Theorem 2.15.)*

***Euclidean Proposition 2.14.*** The sum of the measures of the angles of any triangle is 180.

***Euclidean Proposition 2.15.*** There exists a triangle such that the sum of the measures of the angles of the triangle is 180.

***Euclidean Proposition 2.16.*** The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles. *(Compare to the Exterior Angle Theorem.)*

***Euclidean Proposition 2.17.*** If a point $C$ is not on segment $\overline{AB}$ but on the circle with diameter $\overline{AB}$, then $\angle ACB$ is a right angle.

***Euclidean Proposition 2.18.*** If $\angle ACB$ is a right angle, then $C$ is on the circle with diameter $\overline{AB}$.

***Euclidean Proposition 2.19.*** The perpendicular bisectors of the legs of a right triangle intersect.

***Euclidean Proposition 2.20.*** There exists an acute angle such that every line intersecting and perpendicular to one side of the angle intersects the other side.

***Euclidean Proposition 2.21.*** There exists an acute angle such that every point in the interior of the angle is on a line intersecting both sides of the angle not at the vertex.

***Euclidean Proposition 2.22.*** If $l_1$, $l_2$, $l_3$, $l_4$ are four distinct lines such that $l_1$ is perpendicular to $l_2$, $l_2$ is perpendicular to $l_3$, and $l_3$ is perpendicular to $l_4$, then $l_1$ intersects $l_4$.

***Euclidean Proposition 2.23.*** There exists a rectangle.

***Euclidean Proposition 2.24.*** There exist two lines equidistant from each other.

***Euclidean Proposition 2.25.*** If three angles of a quadrilateral are right angles, then so is the fourth.

***Euclidean Proposition 2.26.*** There exists a pair of similar triangles that are not congruent. *(Two triangles are* similar *if and only if corresponding angles are congruent and the corresponding sides are proportional.)*

***Euclidean Proposition 2.27.*** The diagonals of a Saccheri quadrilateral bisect each other.

***Euclidean Proposition 2.28.*** One of the summit angles of a Saccheri quadrilateral is a right angle.

***Euclidean Proposition 2.29.*** Any three lines have a common transversal.

***Euclidean Proposition 2.30.*** There do not exist three lines such that each two are on the same side of the third.

***Euclidean Proposition 2.31.*** In $\Delta ABC$, if $M$ is the midpoint of segment $\overline{AB}$ and $N$ is the midpoint of segment $\overline{AC}$, then the length of segment $\overline{MN}$ is equal to half the length of segment $\overline{BC}$.