2.8 Euclidean, Hyperbolic, and Elliptic Geometries

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.
—Nikolai Lobachevsky (1793–1856)

Euclidean Parallel Postulate. Through a point not on a line there is exactly one line parallel to the given line.

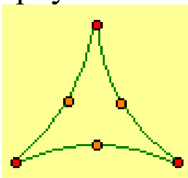
Hyperbolic Parallel Postulate. Through a point not on a line there is more than one line parallel to the given line.

Elliptic Parallel Postulate. Any two lines intersect in at least one point.

Prior to the discovery of non-Euclidean geometries, Euclid's postulates were viewed as absolute truth, not as mere assumptions. Several philosophical questions arose from the discovery of non-Euclidean geometries. What is truth? Which geometry is the correct geometry? Is the physical world Euclidean or non-Euclidean? The view now is that the truth or correctness of the geometry depends on the axiomatic system used.

The third question was investigated by Carl Friedrich Gauss (1777–1855) by measuring the angles formed by a triangle determined by three mountain peaks, to see whether or not the sum of the measures of the angles of a triangle is 180°. Nicolai Lobachevsky (1793–1856) considered astronomical distances for triangles. The errors in their results for the sum of the measures of the angles of a triangle were within expected measurement error. To further complicate the problem, Henri Poincaré (1854–1912) pointed out that when performing an experiment in the physical world, an experimenter also makes assumptions. For example, how can an experimenter know for sure whether light rays travel in a straight line or follow a curved path?

The question still remains: Which geometry should be used when modeling the physical world? The answer depends on what problem is being modeled. Poincaré's view was that the experimenter should choose the geometry that is the most "convenient"; that is, make the choice based on the facts of the experiment in such a manner that contradictions are avoided. Euclidean geometry is generally used in surveying, engineering, architecture, and navigation for short distances; whereas, for large distances over the surface of the globe spherical geometry is used.



Experiments have indicated that binocular vision is hyperbolic in nature. When asked to place light sources at the midpoints of the sides of a triangle determined by three light sources viewed from a horizontal plane, subjects in an experiment usually placed the light sources such that the sides curved inward. (See Ogle's article in *Science* 9 March 1962.) In his study of relativity, Albert Einstein (1879–1955) used a form of Riemannian geometry based on a generalization of elliptic geometry to higher dimensions in which geometric properties vary from point to point. Relativity theory implies that the universe is Euclidean, hyperbolic, or elliptic depending on whether the universe contains an equal, more, or less amount of matter and energy than a certain fixed amount.

Some properties of Euclidean, hyperbolic, and elliptic geometries.		
Euclidean	***Hyperbolic***	***Elliptic***
Two distinct lines intersect in one point.	Two distinct lines intersect in one point.	(single) Two distinct lines intersect in one point. (double) Two distinct lines intersect in two points.
The sum of the measures of the angles of a triangle is 180.	The sum of the measures of the angles of a triangle is less than 180.	The sum of the measures of the angles of a triangle is more than 180.
Similar triangles that are not congruent exist.	Similar triangles are congruent.	Similar triangles are congruent.
Rectangles exist.	No quadrilateral is a rectangle.	No quadrilateral is a rectangle.
A line does not have finite length and is unbounded.	A line does not have finite length.	A line has finite length and is unbounded.
Two parallel lines are equidistant.	No two parallel lines are equidistant.	Parallel lines do not exist.
The summit angles of a Saccheri quadrilateral are right angles.	The summit angles of a Saccheri quadrilateral are acute angles.	The summit angles of a Saccheri quadrilateral are obtuse angles.
Two distinct lines do not enclose a finite area.	Two distinct lines do not enclose a finite area.	Two distinct lines enclose a finite area.
The area of a triangle is equal to half the product of the base and height.	The area of a triangle is proportional to its defect.	The area of a triangle is proportional to its excess.
A unique line perpendicular to a given line through a point not on the line.	A unique line perpendicular to a given line through a point not on the line.	(single) All lines perpendicular to a given line intersect at a point (pole). (double) All lines perpendicular to a given line intersect at two antipodal points.
A line separates a plane.	A line separates a plane.	(single) A line does not separate a plane. (double) A line separates a plane.

Exercise 2.80. Discuss the questions. Is the physical world Euclidean, hyperbolic, or elliptic? How could you prove the answer to the question? Could the physical world be proved to be hyperbolic or elliptic, but not Euclidean?

If only it could be proved...that "there is a Triangle whose angles are together not less than two right angles"! But, alas, that is an* ignis fatuus *that has never yet been caught!
—Charles Lutwidge Dodgson (Lewis Carroll) (1832–1898)