Appendix B - Birkhoff's Axioms for Euclidean Geometry

George D. Birkhoff would be referred to by mathematicians as the author of this web course's "great-great-grandfather" as a direct line of thesis advisors, since Timothy Peil's Ph.D. thesis advisor was Allan Peterson whose advisor was John Barrett whose advisor was Hyman Ettlinger whose advisor was George Birkhoff.

***- from the** Mathematics Genealogy Project*

Timothy Peil authored this web course.

Introductory Note. Birkhoff's Axiom set is an example of what is called a *metric geometry*. A metric geometry has axioms for distance and angle measure, then betweenness and congruence are defined from distance and angle measures and properties of congruence are developed in theorems.

(Some web browsers display some characters incorrectly, an angle shows as ∠ and not equal to shows as ≠.)

Undefined Elements and Relations.

- *points* $A, B, \ldots$
- sets of points called *lines* $l, m, \ldots$
- *distance* between any two points A and B: a nonnegative real number $d(A,B)$ such that $d(A,B) = d(B,A)$
- *angle* formed by three ordered points A, O, B $(A \neq O, B \neq O)$: $\angle AOB$ a real number (mod 2π). The point O is called the vertex of the angle.

Postulate I. (*Postulate of Line Measure*) The points $A, B, \ldots$ of any line l can be placed into one-to-one correspondence with the real numbers x so that $|x_A - x_B| = d(A,B)$ for all points A, B.

Postulate II. (*Point-Line Postulate*) One and only one line l contains two given points P, Q $(P \neq Q)$.

Postulate III. (*Postulate of Angle Measure*) The half-lines $l, m, \ldots$ through any point O can be put into one-to-one correspondence with the real numbers a(mod 2π), so that, if $A \neq O$ and $B \neq O$ are points of l and m, respectively, the difference $a_m - a_l$(mod 2π) is $\angle AOB$. Furthermore if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV. (*Similarity Postulate*) If in two triangles ΔABC, $\Delta A'B'C'$ and for some constant $k > 0$, $d(A',B') = kd(A,B)$, $d(A',C') = kd(A,C)$, and $\angle B'A'C' = \pm\angle BAC$, then also $d(B',C') = kd(B,C)$, $\angle A'B'C' = \pm\angle ABC$, and $\angle A'C'B' = \pm\angle ACB$.

Defined Terms

- A point B is *between* A and C (A ?C), if $d(A,B) + d(B,C) = d(A,C)$.
- The *half-line* l' with *endpoint* O is defined by two points O, A in line l $(A \neq O)$ as the set of all points A' of l such that O is not between A and A'.
- The points A and C, together with all point B between A and C, for *segment* AC.
- If A, B, C are three distinct points, the segments AB, BC, CA are said to form a *triangle* ΔABC with sides AB, BC, CA and *vertices* A, B, C.

Birkhoff, George D., A set of postulates for plane geometry, *Annals of Mathematics* 33: pp. 329-345, 1932.