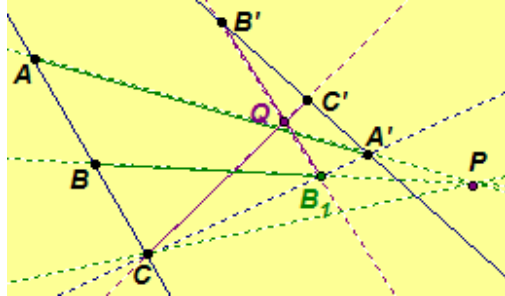


## 4.6.2 Fundamental Theorem of Projective Geometry

*All truths are easy to understand once they are discovered; the point is to discover them..*

—  [Galileo Galilei \(1564–1643\)](#)

In the [introduction](#) to projective geometry, we stated that in a later section we would consider a



mapping between two [pencils of points](#). We begin by showing that there exists a [projectivity](#) between two pencils of points. (*The first video is the lecture and the second video is the construction in Geometer's Sketchpad.*)



Assume  $A, B, C$  are elements of a pencil with axis  $p$  and  $A', B', C'$  are elements of a pencil with axis  $p'$ . Further, assume the points are distinct and the axes  $p$  and  $p'$  are distinct. We desire to find a projectivity so that



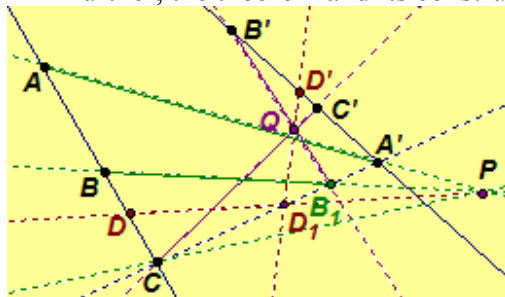
$ABC$  is projectively related to  $A'B'C'$ , i.e.  $ABC \wedge A'B'C'$ . Since a projectivity is a composition of [perspectivities](#), we construct two perspectivities to map  $ABC$  to  $A'B'C'$ .

To construct the first perspectivity, we define the center of a perspectivity that will map  $A$  to  $A'$  and  $C$  to itself with the image of  $B$  to be found. Let  $P$  be a point on  $AA'$  that is distinct from  $A$  and  $A'$ . (*How do we know point  $P$  exists?*) Let  $B_1 = BP \cdot A'C$ . Thus  $ABC \overset{P}{\wedge} A'B_1C$ .

For the second perspectivity, we define the center of a perspectivity that maps  $A'$  to itself,  $B_1$  to  $B'$ , and  $C$  to  $C'$ . Let  $Q = B_1B' \cdot CC'$ . Then  $A'B_1C \overset{Q}{\wedge} A'B'C'$ . Since  $ABC \overset{P}{\wedge} A'B_1C$  and  $A'B_1C \overset{Q}{\wedge} A'B'C'$ , we have  $ABC \wedge A'B'C'$ . We have proven the following theorem.

**Theorem 4.10.** *If  $A, B, C$  and  $A', B', C'$  are distinct elements in pencils of points with distinct axes  $p$  and  $p'$ , respectively, then there exists a projectivity such that  $ABC$  is projectively related to  $A'B'C'$ .*

Further, the theorem and its constructive proof give a procedure to determine a corresponding point  $D'$



on axis  $p'$  by following the perspectivities when a fourth point  $D$  on axis  $p$  is given. That is, let  $D$  be an element of axis  $p$ . First, find  $D_1$  on the pencil of points with  $A', B_1$ , and  $C$  by mapping  $D$  through center  $P$ ; that is, let  $D_1 = DP \cdot A'C$ . Next, map  $D_1$  to  $D'$  by mapping  $D_1$  through the center  $Q$  to  $p'$ , i.e.,  $D' = D_1Q \cdot p'$ . Then  $ABCD \overset{P}{\wedge} A'B_1CD_1$  and  $A'B_1CD_1 \overset{Q}{\wedge} A'B'C'D'$ . Hence,  $ABCD \wedge A'B'C'D'$ .

**Exercise 4.30.** Show that  $P$  in the constructive proof of Theorem 4.10 exists.

**Exercise 4.31.** (a) State and prove the dual of Theorem 4.10. (*Hint: principle of duality*) (b) Assume  $a, b, c$  are elements of a pencil with center  $P$  and  $a', b', c'$  are elements of a pencil with center  $P'$ . Use dynamic geometry software to construct  $abc \wedge a'b'c'$  and find a corresponding line  $d'$  of a line  $d$ . (*Hint: write the dual of the steps of the above construction for points.*)

In the above theorem, we have shown a projectivity exists between two pencils of points with three elements. But, is the projectivity unique? Since an arbitrary point was chosen in the construction, a different point would give different perspectivities. That is, would a different point  $D'' \neq D'$  be determined, if the perspectivities were different? [Click here](#) to investigate the uniqueness of the perspectivity [GeoGebra](#) or [JavaSketchpad](#) constructed in the proof of the previous theorem. Also, to help answer these questions consider [Axiom 6](#).

**Axiom 6.** *If a projectivity on a pencil of points leaves three distinct points of the pencil invariant, it leaves every point of the pencil invariant.*

Axiom 6 implies that a projectivity on a pencil that leaves three elements of the pencil invariant is the identity mapping. What implications does this axiom have for distinct pencils of points? What implications does this axiom have for a projectivity on a pencil of points where no group of three points are mapped to themselves? Can this axiom extend the above theorem for constructing a projectivity between two pencils of points to more than three points? All of these questions are answered by the *Fundamental Theorem of Projective Geometry*, which has the surprising result that only three pairs of points are needed to determine a unique projectivity between two pencils of points.



**Theorem 4.11. (Fundamental Theorem of Projective Geometry)** *A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.*

In other words, if  $A, B, C, D$  are in a pencil of points with axis  $p$  and  $A', B', C'$  are in a pencil of points with axis  $p'$ , then there exists a unique point  $D'$  on  $p'$  such that  $ABCD \wedge A'B'C'D'$ .

*Proof.* Assume  $A, B, C, D$  are in a pencil of points with axis  $p$  and  $A', B', C'$  are in a pencil of points with axis  $p'$ . We have shown that there exists a point  $D'$  on  $p'$  such that  $ABCD \wedge A'B'C'D'$ . Suppose there is a projectivity and a point  $D''$  such that  $ABCD \wedge A'B'C'D''$ . Since  $A'B'C'D' \wedge ABCD$  and  $ABCD \wedge A'B'C'D''$ , we have  $A'B'C'D' \wedge A'B'C'D''$ . Therefore, by Axiom 6,  $D' = D''$ . //

Note the [principle of duality](#) extends the fundamental theorem to pencils of lines. And it can easily be extended to where one set is a pencil of lines and the other is a pencil of points.  $ABCD \wedge A'B'C'D'$ ,  $abcd \wedge ABCD$ , and  $abcd \wedge a'b'c'd'$ .

**Corollary 4.12.** *A projectivity between two distinct pencils of points with a common element that corresponds to itself is a perspectivity.*

**Exercise 4.32.** Given four collinear points  $A, B, C, D$ , use a dynamic geometry software to construct each projectivity and the image of the fourth point  $D$ . (For at least one of the exercises, you may need to draw another arbitrary pencil of points that is distinct from the given pencil of points.)

- $ABC \wedge ACD$
- $ABC \wedge BDA$
- $ABC \wedge BAC$

**Exercise 4.33.** Given four concurrent lines  $a, b, c, d$ , use a dynamic geometry software to construct each projectivity and the image of the fourth line  $d$ .

- $abc \wedge acd$
- $abc \wedge bda$
- $abc \wedge bac$

**Exercise 4.34.** Prove: Given three concurrent lines  $a, b$ , and  $c$  and two points  $P$  and  $Q$  not on any of the three lines. If  $A_p$  and  $B_p$  are points on  $a$  and  $b$ , respectively, such that  $A_pP \cdot B_pQ = C_p$  is on line  $c$ , then  $A_p \wedge B_p$ .

**Exercise 4.35.** Prove Corollary 4.12. (Remember a perspectivity is a projectivity, but a projectivity need not be a perspectivity.)

[4.6.1 Definition of Perspectivity and Projectivity](#)  [4.6.3 Harmonic Sets and Projectivity](#)