4.6.3 Harmonic Sets and Projectivity

If a man is at once acquainted with the geometrical foundation of things and with their festal splendor, his poetry is exact and his arithmetic musical.
—Ralph Waldo Emerson (1803–1882)

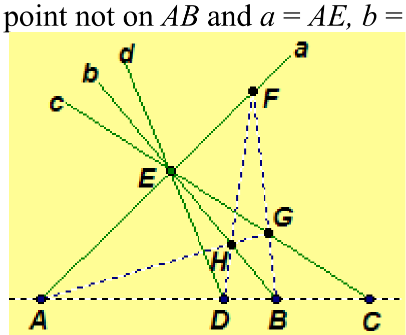
The Fundamental Theorem of Projective Geometry states that three pairs of corresponding points determine a projectivity between two pencils of points. But, a harmonic set of points consists of four points. Several questions arise.

- Is the projection of a harmonic set, also a harmonic set?
- That is, is a harmonic relationship invariant under a projectivity?
- Does a projectivity exist between two harmonic sets?

In this section, we examine these questions.

We begin by investigating the first two questions. Since a projectivity is a finite product of elementary correspondences, we first show that an elementary correspondence preserves a harmonic relation.

Let *H*(*AB,CD*) be a harmonic set of points. Let *a,b,c,d* be a pencil of lines with center *E* such that *E* is a point not on *AB* and *a* = *AE, b* = *BE, c* = *CE, d* = *DE.* Hence, we have an elementary correspondence



$ABCD \barwedge abcd$. We assert that *H*(*ab,cd*), a harmonic set of lines.

Since *H*(*AB,CD*), by the constructive proof of Theorem 4.6, there is a complete quadrangle *EFGH* such that *A* and *B* are diagonal points with *A* = *EF* · *GH, B* = *FG* · *EH, C* = *GE* · *AB*, and *D* = *FH* · *AB.* We desire a complete quadrilateral such that *a* and *b* are diagonal lines and *c* and *d* are determined from the third diagonal line. Since *FG* · *FH* = *F* and *AH* · *AB* = *A* are on *a*, we would have line *a* as a diagonal line for the complete quadrilateral determined by *FG, FH, AH,* and *AB. (Show this is a complete quadrilateral.)* Thus, we have *FG* · *FH* = *F* and *AH* · *AB* = *A* on *a*, *FG* · *AB* = *B* and *FH* · *AH* = *H* on *b*, *FG* · *AH* = *G* on *c*, and *FH* · *AB* = *D* on *d.* Hence, by definition of a harmonic set of lines, *H*(*ab,cd*). Further, by the principle of duality, the converse is also true.

Therefore, an elementary correspondence maps a harmonic set of points/lines to a harmonic set of lines/points. Since a projectivity is a finite product of elementary correspondences, a projectivity maps a harmonic set to another harmonic set. We have proven that *a harmonic relationship is invariant under a projectivity* as stated in the following theorem.

Theorem 4.13. If H(AB,CD) and $ABCD \wedge A'B'C'D'$***, then H(A'B',C'D').***

Exercise 4.36. Show the complete quadrilateral defined by *FG*, *FH*, *AH*, and *AB* in the above proof is in fact a complete quadrilateral.

The above result, together with the Fundamental Theorem of Projective Geometry and Corollary 4.9, answers our other questions about the relationship between harmonic sets.

Theorem 4.14. There exists a projectivity between any two harmonic sets.

Exercise 4.37. Prove Theorem 4.14.

If all art aspires to the condition of music, all the sciences aspire to the condition of mathematics.
—George Santayana (1863–1952)