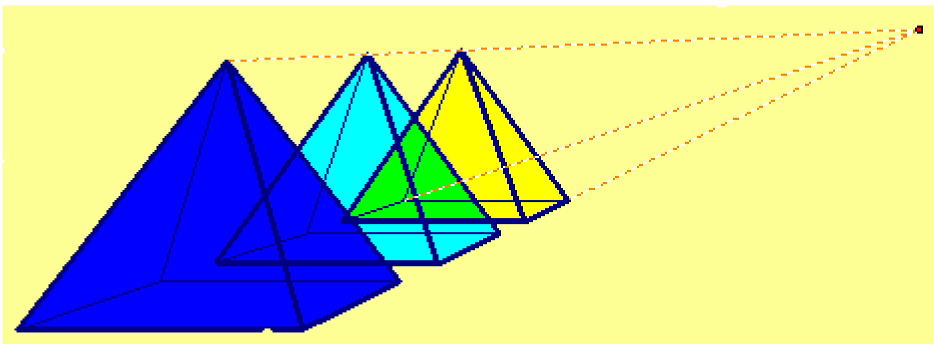
# 4.1.1 Introduction to Projective Geometry

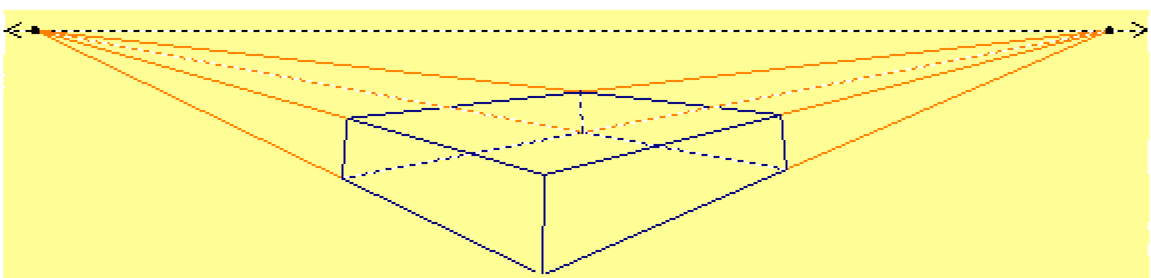
***And since geometry is the right foundation of all painting, I have decided to teach its rudiments and principles to all youngsters eager for art...***
***—Albrecht Dürer (1471–1528)***

Much of the motivation for the study of projective geometry comes from art. How should one draw a three dimensional object in two dimensions and maintain a sense of depth? Go to a local art museum or visit an internet art museum such as the National Gallery of Art, The Metropolitan Museum of Art, or Art on the Net. Examine how different artists have given depth to their pictures.

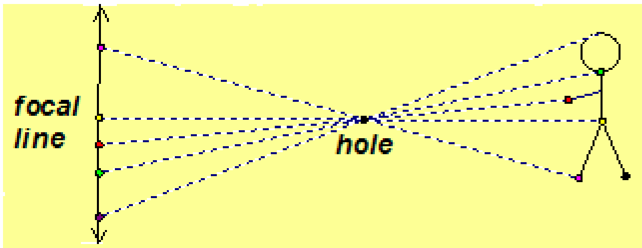
One-point perspective is used to draw the three pyramids in the following diagram. Click here for a dynamic illustration of the pyramids GeoGebra or JavaSketchpad.



In one-point perspective, all observation lines intersect at an *ideal point* along a *horizon line*. The concept grew out of the artistic view that parallel lines will intersect at ideal points on the horizon. Consider looking down railroad tracks and how the tracks appear to converge in the distance. The next figure is an example of a box drawn with two-point perspective. The view is like standing near a street corner, then looking down each side. Here, there are two ideal points along the horizon line. Click here for a dynamic illustration of two-point perspective GeoGebra or JavaSketchpad.

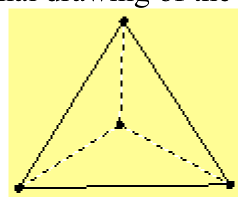


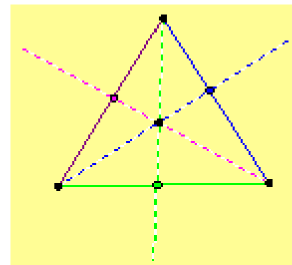
The projection of a figure through a pin hole is often used as an inexpensive and safe method for viewing a solar eclipse. In our two dimensional illustration (below), the stick figure has been projected through a point onto the line. Notice how the image is inverted through the pin-hole. This is an illustration of what will be called a *perspectivity* between two *pencils* of points. The image of the person is projected onto the focal line through the pin-hole. Click here for a dynamic illustration of the view through a pin hole GeoGebra or JavaSketchpad.



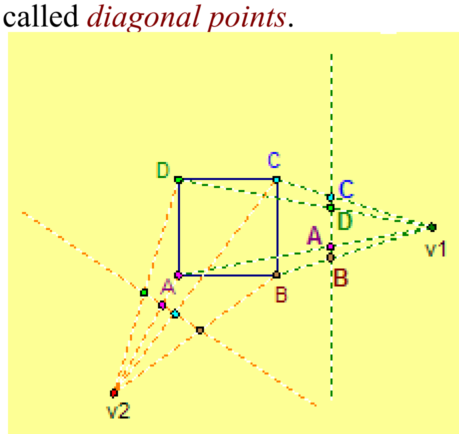

Notice that the lines all intersect. One of the axioms for projective geometry requires that any two distinct lines intersect in at least one point. Hence, projective geometry is a non-Euclidean geometry.

Consider a tetrahedron drawn in a plane. The 2-dimensional drawing of the tetrahedron consists of four points where no three of the points are collinear. This motivates one of the axioms for projective geometry. The axiom will be stated as *"There exist at least four points, no three of which are collinear."* The figure formed by the four points and the six lines determined by those points will be called a *quadrangle*.





Further, if we extend the sides of the 2-dimensional drawing of a tetrahedron, we note that each pair of "opposite sides" intersect at a point. (See the colored correspondence in the diagram.) These three points formed by the intersections of the opposite sides are not collinear, which is a motivation for an axiom stated as *"The three diagonal points of a complete quadrangle are never collinear,"* where the points of intersection of the opposite sides are called *diagonal points*.




Consider a square being viewed from the side by two viewers, *v*1 and *v*2, from different perspectives as shown in the given figure. Each viewer would see the vertices of the square along a line. The perspective of viewer *v*1, of the vertices of the square, is the points along a line in the order *C, D, A, B*; whereas, the perspective of viewer *v*2, of the vertices of the square, is the points along a line in the order *D, A, C, B.* How would the order of the vertices change, if a viewer moved to obtain a different perspective? ***To check your conjectures go to investigate with a dynamic illustration [GeoGebra](GeoGebra)*** or ***[JavaSketchpad](JavaSketchpad).***

The points along the line from which the viewer sees the vertices of the square will be defined in a later section as a *pencil of points*. Each diagram of a viewer's perspective (observation point, view lines, and pencil of points) will be called a *perspectivity* with the point representing the viewer called the *center of the perspectivity*.

In the section on perspectivities and projectivities, we will study relationships between what is perceived (pencil of points) by the two viewers. How can the perspective of one viewer be projected onto the perspective of the other? This leads to the more general concept of *projectivity,* which will be defined as a product of *perspectivities*.

4.1.2 Historical Overview

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