Session 11 – Division, Properties of Equality and Equation Solving

How does the operation of division used in the two following problems relate to sets?

> *An employer distributes $18,300 equally in bonuses to fifteen employees. How much money does each employee receive?*
>
> *Each employee received $720 when an employer distributed $15,840 equally in bonuses. How many employees received bonuses?*

Division is used to find the answer to both types of questions. These two problems illustrate two different views or models for division of whole numbers.

In the first problem, we know the number of sets (number of employees) and need to find the number of elements for each set (number of dollars each employee receives). The problem “divides” the amount ($18,300) among 15 equivalent sets and asks the question: “How many elements should each set receive?” This concept of sharing objects equally among several sets is called the *partition model* (some call it the *partitive model*) because a set of objects is partitioned into several equal sized sets.

In the second problem, we know the number of elements in each set (number of dollars each employee received) and need to find how many sets there are (number of employees). The problem “divides” the amount ($15,840) into sets of $720 and asks the question: “How many sets can be made?” This concept of finding the number of set that can receive the same number of objects is called the *repeated-subtraction model* (sometimes called the *measurement model*) because we may find the solution by finding the number of times the amount can be subtracted from the original.

For the first problem if each of the fifteen employees received $1,220 all the $18,300 would be distributed. Stating the result this way, we note that $15 \cdot 1220 = 18300$. In the second problem, after subtracting $720 twenty-two times, all the $15,840 would be distributed. Again, stating the result this way, we note that $22 \cdot 720 = 15,840$; the inverse of repeated addition. This relationship from both models leads to division as the inverse operation of multiplication. That is,

$$a \div b = c \text{ only when } b \cdot c = a.$$

We use this inverse relationship to check division answers and also to solve multiplication and division equations.

More Examples

Identify the sets and the model used in each division problem.

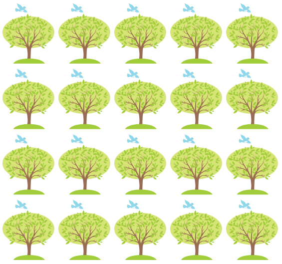
Example: Three children are to share 12 pencils equally, how many pencils should each child receive?

Each child is the recipient of a set and the pencils are the elements. The question may be rewritten as “How many elements should each set receive?” This is the partition model.

Note that $12 \div 3 = 4$ and $3(4) = 12$. Each child would receive four pencils.

Example: Twenty-one trees are to be planted in four equal rows, how many trees are to be planted in each row?

Each row represents a set and the trees are the elements. The question may be rewritten as “How many elements are to be put into each set?” This is the partition model.



Note that $21 \div 4 = 5$ R1 and $4(5) + 1 = 21$. Five trees would be planted in each row with one tree left over.

Example: Each child is to receive 3 pencils from a box containing 12 pencils. How many children will get pencils?

Each child is the recipient of a set and the elements are the pencils. The question may be rewritten as “How many sets will get three elements?” This is the repeated-subtraction model.

Note that $12 - 3 - 3 - 3 - 3 = 0$, so $12 \div 3 = 4$ and $4(3) = 12$.

Four children will receive pencils.

Example: A car is at mile marker 200 on I-94. If the car is traveling west at 50 mph, how long will it take the car to reach the state border?

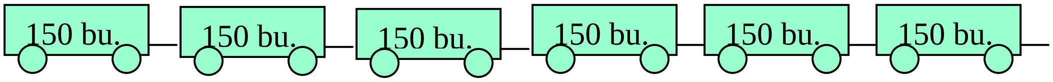
Each hour can be associated with a set and the elements are the miles. The question may be rewritten as “How many sets will take 50 elements?” This is the repeated-subtraction model.

Note that $200 - 50 - 50 - 50 - 50 = 0$, so $200 \div 50 = 4$ and $4(50) = 200$.

It will take the car four hours to reach the border.

Example: A wagon will hold 150 bushels of wheat. How many wagon loads will it take to fill a bin that holds 900 bushels?

Each wagon is a set and the elements are the bushels. The question may be rewritten as “How many sets will hold 150 elements?” This is the repeated-subtraction model.



Note that $900 - 150 - 150 - 150 - 150 - 150 - 150 = 0$,

so $900 \div 150 = 6$ and $6(150) = 900$. Six wagon loads are needed to fill the bin.

Notation and Vocabulary for Division

Missing-factor Definition for Division: The *missing-factor definition for division* is based on the idea of division being the inverse operation of multiplication. Let a and b be any whole numbers with $b \neq 0$. Then $a \div b = c$ if and only if $bc = a$.

The whole number a is the *dividend*, b is the *divisor*, and c is the *quotient* (the solution).

Other notations: In each the division problem below, a is the *dividend* (the number that is being divided); b is the *divisor* (the number we are dividing by); and c is the *quotient* **(**the answer to the division problem).

$$b\overset{c}{\overline{\smash{\big)}a}} \quad \text{or} \quad a \div b = c \quad \text{or} \quad \frac{a}{b} = c$$

Example: A class of twenty-four students is to breakup into groups of three. How many groups would be formed?

$$3\overset{8}{\overline{\smash{\big)}24}} \quad \text{or} \quad 24 \div 3 = 8 \quad \text{or} \quad \frac{24}{3} = 8$$

Eight groups of three would be formed by the class of twenty-four.

Division with Zero

Many people have trouble with problems involving division where either the dividend or divisor is zero. Note that in the missing-factor definition, we assume that the divisor cannot be zero. Why do we make that assumption?

What would happen if the divisor was 0?

Example: $5 \div 0$.

Rewrite the problem $5 \div 0 = c$ as a multiplication problem, $0 \cdot c = 5$. But, we know from multiplication of whole numbers that 0 times any whole number is 0, which means there is no whole number solution to the problem.

Example: $0 \div 0$.

Rewrite the problem $0 \div 0 = c$ as a multiplication problem, $0 \cdot c = 0$. Again, we know that 0 times any whole number is 0, which means that every whole number is a solution to the problem; e.g., $0 \cdot 4 = 0$, $0 \cdot 35 = 0$, etc.

The above two examples illustrate why we do not define division by zero. When a problem involves division by zero (0 as the divisor), we state that the solution is *undefined* (or that there is no solution).

What about when the dividend is 0?

Example: $0 \div 6$.

Rewrite the problem $0 \div 6 = c$ as a multiplication problem, $6 \cdot c = 0$. We know from multiplication that this solution must be 0 since $6 \cdot 0 = 0$ and multiplication by a nonzero value would give a nonzero answer; e.g., $6 \cdot 1 = 6$, $6 \cdot 7 = 42$, etc.

The example illustrates that $0 \div b = 0$ whenever $b \neq 0$.

Algorithms for Division of Whole Numbers

Division requires a good understanding of place value, because the standard algorithm for division is based on place values.

Compare these related multiplication and division problems:

$2 \times 4 = 8$ and $\begin{array}{r} 4 \\ 2\overline{)8} \end{array}$		
$2 \times 40 = 80$ and $\begin{array}{r} 40 \\ 2\overline{)80} \end{array}$		
$2 \times 400 = 800$ and $\begin{array}{r} 400 \\ 2\overline{)800} \end{array}$		

If we are careful about how we line up the digits in the quotient with the digits in the dividend, we see how the three division problems are related.

When we do the division for each of the above three problems, we often say that “2 goes into 8 four times” and place the 4 above the 8. The zeros are added to indicate the proper place-value of the 4. We illustrate this with the money model for each of the problems, which show that we are forming two sets of bills and answering the question: “How many elements are in each set?”

Scaffold Algorithm for Division

Example: May and Jay’s are to share an inheritance of $860. How much should each receive? To solve the problem, we want to divide 860 by 2.

$$\begin{array}{r} 400 \\ 2\overline{)860} \\ -800 \\ \hline 60 \end{array}$$

First, we begin by dividing 8 hundred by 2, which gives us 4 hundred. Since we have used 8 hundred, we subtract 860 – 800 = 60. In the model, we still have $60 to share.

$$\begin{array}{r} 30 \\ 400 \\ 2\overline{)860} \\ -800 \\ \hline 60 \\ -\ \ 60 \\ \hline 0 \end{array}$$

Next, we share the $60. Since 2 × 30 = 60, we need to add another 30 to our quotient. Notice we place the 30 in the proper place value above the 400. We have also subtracted 60 – 60 = 0. In the model, we have no money left to share.

$$\begin{array}{r}
\left.\begin{array}{r}30\\ 400\end{array}\right\}=430\\
2\overline{)860}\\
-800\\
\hline
60\\
-\quad 60\\
\hline
0
\end{array}$$

The last step is to sum the two partial quotients to obtain the final quotient. This method of division is called the ***scaffold algorithm*** for long division.

May and Jay would each receive $430.

Here are two different examples that use the scaffold algorithm to divide 976 by 2. The first example uses the most efficient partial quotients. The second example uses more partial quotients but they are in smaller pieces; this is like passing out a large number of items by giving each person a few at a time. Since we may pass out any number of items at a time, the number of partial quotients we use does not matter. We do need to pay attention to the place values.

$$\begin{array}{rl}
\left.\begin{array}{r}3\\ 90\\ 400\end{array}\right\}=\textcircled{493} & \\
2\overline{)986} & \\
-\ 800 & \dashleftarrow\text{- - - - -}\ 2\times 400\\
\hline
186 & \\
-\ 180 & \dashleftarrow\text{- - - - -}\ 2\times 90\\
\hline
6 & \\
-\quad 6 & \dashleftarrow\text{- - - - -}\ 2\ \times 3\\
\hline
0 &
\end{array}$$

$$\begin{array}{rl}
\left.\begin{array}{r}3\\ 40\\ 50\\ 200\\ 200\end{array}\right\}=\textcircled{493} & \\
2\overline{)986} & \\
-\ 400 & \dashleftarrow\text{- - - - -}\ 2\times 200\\
\hline
586 & \\
-\ 400 & \dashleftarrow\text{- - - - -}\ 2\times 200\\
\hline
186 & \\
-\ 100 & \dashleftarrow\text{- - - - -}\ 2\times 50\\
\hline
86 & \\
-\quad 80 & \dashleftarrow\text{- - - - -}\ 2\times 40\\
\hline
6 & \\
-\quad 6 & \dashleftarrow\text{- - - - -}\ 2\times 3\\
\hline
0 &
\end{array}$$

Many students, who find the standard algorithm for long-division difficult, find the scaffold method helpful, especially when they use “comfortable chunks” instead of always looking for the most efficient partial quotient. If long division is difficult for you, try using the scaffold method.

Example:
Here are three different ways to approach 1008 ÷18.

Scaffold

$$\begin{array}{r}
\left.\begin{array}{r}1\\ 5\\ 10\\ 20\\ 20\end{array}\right\}=56\\
18\overline{)1008}\\
-360\\
\hline
648\\
-\quad 360\\
\hline
288\\
-\quad 180\\
\hline
108\\
-\quad 90\\
\hline
18\\
-\quad 18\\
\hline
0
\end{array}$$

Most Efficient
Scaffold

$$\begin{array}{r}
\left.\begin{array}{r}6\\ 50\end{array}\right\}=56\\
18\overline{)1008}\\
-\quad 900\\
\hline
108\\
-\quad 108\\
\hline
0
\end{array}$$

Standard
Algorithm

$$\begin{array}{r}
56\\
18\overline{)1008}\\
-\quad 900\\
\hline
108\\
-\quad 108\\
\hline
0
\end{array}$$

Solution to an Equation

Remember that a value is a solution to an equation when that value is substituted in for the variable in the equation, the resulting arithmetic statement is a true statement.

Example: Is 6 a solution to the equation $42 \div x = 7$?
Yes, since $42 \div 6 = 7$ is a true statement, 6 is a solution to this equation.
We state that 6 is a solution to this equation by writing $x = 6$.
We write the solution set as $\{6\}$.

Example: The value 4 is *not* a solution to the equation $10 \div x = 5$ since $10 \div 4 = 2$ R.$2 \neq 5$.

Example: The value 12 is a solution to $5x = 60$ since $5 \cdot 12 = 60$.
The solution set for the equation is $\{12\}$.

Multiplication Property of Equality: If we multiply both sides of an equation by the same *non-zero* value, the resulting statement is still an equation and it has the same solution set as the original equation.

General Property: If $a = b$ and $c \neq 0$, then $ac = bc$.

Note what happens when we use this property with the missing-factor definition for division.

$$\begin{aligned} a \div b &= c & & \\ b(a \div b) &= bc & & \text{Multiplication Property of Equality} \\ &= a & & b \cdot c = a. \end{aligned}$$

The last step uses the fact that $a \div b = c$ means $b \cdot c = a$ and substitution. This gives a useful cancellation property for solving equations.

Cancellation Properties: The Cancellation Property for Multiplication and Division of Whole Numbers says that if a value is multiplied and divided by the same nonzero number, the result is the original value.

General Properties: $b(a \div b) = a$ and $(a \cdot b) \div b = a$

$$b \cdot \frac{a}{b} = a \quad \text{and} \quad \frac{ab}{b} = a$$

Note that in the following example, we use the algebraic form of the division sign. Also, note the first step uses the Multiplication Property of Equality and the second step uses a cancellation property.

Example:

$$\begin{aligned} \frac{x}{6} &= 9 \\ 6\left(\frac{x}{6}\right) &= 6(9) \\ x &= 54 \end{aligned}$$

Check:

Is $\frac{54}{6} = 9$?

Yes, $\begin{array}{r} 9 \\ 6\overline{)54} \end{array}$

Division Property of Equality: If we divide both sides of an equation by the same *non-zero* value, the resulting statement is still an equation and it has the same solution set as the original equation.

General Property: If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.

Challenge Problem: Can you show the second cancellation property? That is show $(a \cdot b) \div b = a$.

Once again, we use the algebraic form of the division sign and a cancellation property.

Example:

$$9x = 270$$
$$\frac{9x}{9} = \frac{270}{9}$$
$$x = 30$$

Check:

Is 9(30) = 270?

Yes.

More on Properties

A property in mathematics must be true for the entire universe under discussion. A proposed property can be shown to *not* be true by giving a counterexample.

Counterexample: A *counterexample* is an example for which a proposed property does not work.

We only need ***one*** counterexample to prove that a proposed property is not true.

Example: Is division of whole numbers commutative?

> We compare $10 \div 2$ and $2 \div 10$. Since $10 \div 2 = 5$ and $2 \div 10 = 0.2$ (decimal fractions will be covered later in the course) and $5 \neq 0.2$, the two expressions are not equal, $10 \div 2 \neq 2 \div 10$.Therefore, we have made a counterexample to show that division is *not* commutative.

Can you find a counterexample that shows that division is not associative?